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ON THE D_α SPECTRUM OF CONNECTED GRAPHS

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ABSTRACT. Let G be a connected graph with $\alpha \in [0,1]$, the D_α-spectral radius of G is defined to be the spectral radius of the matrix $D_\alpha(G)$, defined as $D_\alpha(G) = \alpha T(G) + (1-\alpha) D(G)$, where $T(G)$ is a transmission diagonal matrix of G and $D(G)$ denotes the distance matrix of G. In this paper, we give some sharp upper and lower bounds for the D_α-spectral radius with respect to different graph parameters.

1. Introduction

Throughout this paper we consider simple connected undirected graphs. Let G be a graph of order n with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, we define the *distance* between vertex u and v in G, denoted by $d_G(u,v)$ or simply d_{uv}, is the length of a shortest path from vertex u to vertex v in G. The maximum distance between any two vertices in G is called *diameter* of G, denoted by $d(G)$. The *distance matrix* of G is the $n \times n$ matrix $D(G) = (d_G(u,v))_{u,v \in V(G)}$. For $u \in V(G)$, the *transmission* of u (also known as the degree distance of u) in G, denoted by T_u, is defined as the sum of distances from u to all other vertices of G, i.e., $T_u(G) = \sum_{v \in V(G)} d_G(u,v)$. The *transmission matrix* $T(G)$ of G is the diagonal matrix of transmissions of G. We define the *distance signless Laplacian* matrix of any graph G as $D_Q(G) = T(G) + D(G)$ and *distance Laplacian* matrix as $D_L(G) = T(G) - D(G)$, where $D(G)$ denotes the distance matrix of G and $T(G)$ the transmission matrix of G.

The distance eigenvalues and especially the distance spectral radius have been extensively studied for many years, see the recent survey [1] and references therein. The distance Laplacian and distance signless Laplacian spectrum of graphs have also received much attention in recent years, especially the problems related to their spectral radius, see [2, 5, 7, 9, 10, 11, 15].

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Aouchiche and Hansen [2] showed that the distance Laplacian eigenvalues and distance signless Laplacian eigenvalues do not decrease when an edge is deleted. In [3], the same authors proved that the star is a unique tree with a minimum distance Laplacian spectral radius. In [6], Alhevaz et al. gave some upper and lower bounds on distance signless Laplacian spectral radius and also determined the distance signless Laplacian spectrum of some graph operations. For more review about distance Laplacian and distance signless laplacian see [2, 5, 7, 9, 10, 11, 12, 15].

In [8], Nikiforov proposed to study the convex linear combinations of the adjacency matrix $A(G)$ of G and diagonal matrix of its vertices $D_{deg}(G)$, i.e.,

$$A_{\alpha(G)} = \alpha A(G) + (1-\alpha)D_{deg}(G), \qquad \alpha \in [0,1],$$

where $A_\alpha(G)$ is called *generalized adjacency matrix* or A_α matrix of G. This concept of generalized adjacency matrix helped in merging the adjacency spectral and signless Laplacian spectral theories. Similarly Cui, He and Tian [13], introduced the generalized distance matrix $D_\alpha(G)$ as the convex combination of $T(G)$ and $D(G)$.

$$D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G), \qquad \alpha \in [0,1],$$

where $D(G)$ denotes the distance matrix of graph G and $T(G)$ denotes the transmission matrix of graph G. Obviously, $D_0(G) = D(G)$ for $\alpha = 0$, which represents the distance matrix of G and $2D_{1/2}(G) = D_Q(G)$, a distance signless laplacian matrix of graph G. We call the eigenvalues of distance matrix $D(G)$ as the distance eigenvalues and denoted them by $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$ while the eigenvalues of $D_\alpha(G)$ matrix of G, we call them distance α-eigenvalues of G. As $D_\alpha(G)$ is a symmetric matrix, the distance α-eigenvalues of G are all real and we denote them by $\lambda_\alpha^{(1)}(G), \ldots, \lambda_\alpha^{(n)}(G)$, arranged in nonincreasing order, where $n = |V(G)|$. The largest distance α-eigenvalue $\lambda_\alpha^{(1)}(G)$ of G is called the *D_α-spectral radius* of G, written as $\rho_\alpha(G)$ and the minimum distance alpha eigenvalue λ_α^n as $\rho_{min}(G)$. Obviously, $\lambda_0^{(1)}(G), \ldots, \lambda_0^{(n)}(G)$ are the distance eigenvalues of G, and $2\lambda_{1/2}^{(1)}(G), \ldots, 2\lambda_{1/2}^{(n)}(G)$ are the distance signless Laplacian eigenvalues of G. Particularly, $\rho_0(G)$ is the distance spectral radius and $2\lambda_{1/2}(G)$ is just the distance signless Laplacian spectral radius of G. For more details on D_α matrix, readers are suggested to see [13, 14, 16, 17]. In this paper we give sharp upper and lower bounds, involving transmission, second transmission and independence number for the distance α-spectral radius of connected graphs, for example the upper bound

$$\rho_\alpha(G) \leqslant \max_{1 \leqslant u \leqslant n} \frac{\alpha T_u + \sqrt{4T_u m_u(1-\alpha) + (\alpha T_u)^2}}{2}$$

(see Theorem 3.3) and lower bound

$$\rho_\alpha(G) \geqslant \frac{1}{\sum_{i=1}^{n} T_i^2} \sum_{u,v\in V(G)} d_{uv}(\alpha(T_u - T_v)^2 + 2T_uT_v)$$

(see Theorem 3.6). In section 2, we briefly introduce the preliminaries. Section 3 comprises the main results of the paper. In section 3, we give some sharp upper and lower bounds for D_α-spectral radius and briefly discussed the equality conditions for each bound. We also state and prove bounds for D_α-spectral radius of the connected bipartite graph and show that graph $K_{1,n-1}$ is a unique graph which maximizes the D_α spectral radius among all graphs with diameter 2. Moreover, we also give some results and bounds on D_α spectral radius of a graph in terms of its connected complement.

2. Preliminaries

Let G be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and a column vector $x = (x_{v_1}, \ldots, x_{v_n})^\top \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$, which maps vertex v_i to x_{v_i}, i.e., $x(v_i) = x_{v_i}$ for $i = 1, \ldots, n$. Then

$$x^T D_\alpha(G)x = \alpha \sum_{u\in V(G)} T_u x_u^2 + 2 \sum_{\{u,v\}\in V(G)} (1-\alpha)d_{uv}x_u x_v, \tag{2.1}$$

or equivalently,

$$x^T D_\alpha(G)x = \sum_{\{u,v\}\in V(G)} d_{uv}(\alpha(x_u^2 + x_v^2) + 2(1-\alpha)x_u x_v). \tag{2.2}$$

Here we recall that a matrix is *irreducible* if it cannot be placed into block upper-triangular form by simultaneous row/column permutations. Since $D_\alpha(G)$ is a nonnegative irreducible matrix, by Perron-Frobenius theorem, $\rho_\alpha(G)$ is simple and there is a unique positive unit eigenvector corresponding to $\rho_\alpha(G)$, which is called the D_α-*Perron vector* of G. If x is the distance α-Perron vector of G, then for each $u \in V(G)$,

$$\rho_\alpha(G)x_u = \alpha T_u x_u + (1-\alpha) \sum_{v\in V(G)} d_{uv}x_v, \tag{2.3}$$

or equivalently,

$$\rho_\alpha(G)x_u = \sum_{v\in V(G)} d_{uv}(\alpha x_u + (1-\alpha)x_v),$$

which is called the α-*eigenequation* of G at vertex u. For a unit column vector $x \in \mathbb{R}^n$ with at least one nonnegative entry, by Rayleigh's principle, we have $\rho_\alpha(G) \geq x^\top D_\alpha(G)x$ with equality if and only if x is the distance α-Perron vector of G.

We denote by $\xi(G)$, the sum of distances between all unordered pairs of vertices in G i.e.,

$$\xi(G) = \frac{1}{2} \sum_{v \in V(G)} T_v.$$

For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then *second transmission* of vertex v is denoted by $\hat{T_v}$ i.e.,

$$\hat{T_v} = \sum_{u=v_1}^{v_n} d_{uv} T_u.$$

A graph is said to be *transmission regular* if T_v is a constant for each $v \in V(G)$. It is clear that any *vertex-transitive graph* (a graph G in which for every two vertices u and v, there exist an automorphism f on G such that $f(u) = v$) is a transmission regular graph. Indeed, the graph on 9 vertices shown in Figure 1 is 14-transmission regular graph but not degree regular and therefore not vertex-transitive. For more examples of transmission regular but not degree regular graphs see [4].

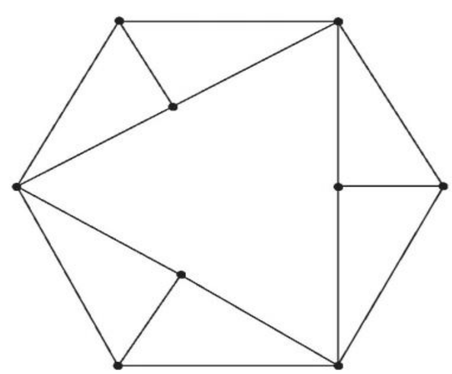

FIGURE 1. The transmission regular but not degree regular graph with the smallest order.

In [17], H. Guo and B. Zhou studied some graph transformation properties and its effect on D_α matrix of connected graph. The deletion or addition of an edge mainly effects the D_α spectral radius.

Lemma 2.1 (Guo and Zhou, [17]). *Let G be a connected graph with $u, v \in V(G)$. If u and v are not adjacent, then $\rho_\alpha(G + uv) \leqslant \rho_\alpha(G)$.*

By [16] it is known that D_α spectrum of K_n is $\{n-1, (\alpha n-1)^{n-1}\}$. Thus:

Theorem 2.2 (Guo and Zhou, [17]). *Let G be a connected graph of order n. Then*

$$\rho_\alpha(G) \geq n - 1,$$

with equality if and only if $G \cong K_n$.

Lemma 2.3 (Díaz, Pastén and Rojo, [14])**.** *Let G be a connected graph on $n \geqslant 2$ vertices and $\alpha \in [1/2, 1]$. Then*

$$\lambda_\alpha^1(G) \geqslant \lambda_\alpha^1(K_n) = n-1 \qquad \text{and}$$
$$\lambda_\alpha^i(G) \geqslant \lambda_\alpha^i(K_n) = \alpha n - 1, \qquad \forall\ 2 \leqslant i \leqslant n.$$

3. Main results

In this section, we give sharp upper and lower bounds on $\rho_\alpha(G)$ for a connected graph G of order n. We first recall some results from [13] and compare the bounds.

Lemma 3.1 (Cui, He and Tian, [13])**.** *Let G be a simple connected graph of order n with transmission sequence $(T_1, \dots, T_n)$ and*

$$\xi(G) = \frac{1}{2} \sum_{v \in V(G)} T_v.$$

Then

$$\rho_\alpha(G) \geqslant \frac{2\ \xi(G)}{n},$$

with equality if and only if G is transmission regular.

Proposition 3.2 (Cui, He and Tian, [13])**.** *Let G be a connected graph and $\{T_1, T_2, \dots, T_n\}$, $\{\hat{T}_1, \hat{T}_2, \dots, \hat{T}_n\}$ be the transmission and second transmission sequence respectively of graph G and $\alpha \in [1/2, 1]$. Then*

$$\rho_\alpha(G) \geqslant \sqrt{\frac{\sum\limits_{i=1}^{n} (\alpha T_i^2 - (\alpha-1)\hat{T}_i)^2}{\sum\limits_{i=1}^{n} T_i^2}}. \tag{3.1}$$

Moreover, if G is transmission regular then equality holds.

Cui et al. in [13] stated the above two results but here we show that the lower bound for $\rho_\alpha(G)$ given in Proposition 3.2 is always better than the bound given in Lemma 3.1. First its obvious that $\sum_{i=1}^n \hat{T}_i = \sum_{i=1}^n T_i^2$. By Cauchy-Shwartz inequality, we see that $(\sum_{i=1}^n \hat{T}_i)^2 \leqslant n \sum_{i=1}^n \hat{T}_i^{\,2}$ and $(\sum_{i=1}^n T_i)^2 \leqslant n \sum_{i=1}^n T_i^2$. Now from Proposition 3.2 we have,

$$\rho_\alpha(G) \geqslant \sqrt{\frac{\sum\limits_{i=1}^{n} (\alpha T_i^2 - (\alpha-1)\hat{T}_i)^2}{\sum\limits_{i=1}^{n} T_i^2}},$$

$$\rho_\alpha(G) > \sqrt{\frac{\sum\limits_{i=1}^{n} (\alpha T_i^2 - (\alpha-1)\hat{T}_i)^2}{n\sum\limits_{i=1}^{n} T_i^2}},$$

$$\rho_\alpha(G) > \sqrt{\frac{(\sum\limits_{i=1}^{n} T_i^2)^2}{n\sum\limits_{i=1}^{n} T_i^2}} = \sqrt{\frac{(\sum\limits_{i=1}^{n} T_i)^2}{n}},$$

$$\rho_\alpha(G) > \sqrt{\frac{(\sum\limits_{i=1}^{n} T_i)^2}{n^2}} = \frac{2\ \xi(G)}{n}.$$

In the following Theorems, we give some new sharp upper and lower bounds for D_α-spectral radius of connected graphs.

Theorem 3.3. *Let G be a simple connected graph with n vertices and the transmission sequence $(T_1, \ldots, T_n)$ and $\alpha \in [1/2, 1]$. Then*

$$\rho_\alpha(G) \leqslant \max_{1\leqslant u\leqslant n} \frac{\alpha T_u + \sqrt{4T_u m_u(1-\alpha) + (\alpha T_u)^2}}{2}, \tag{3.2}$$

where $m_u = (1/T_u)\sum_{v=1}^{n} d_{uv}T_v$. Moreover, if G is a transmission regular graph then equality holds.

Proof. Let $x = (x_1, \ldots, x_n)^T$ be the distance α-Perron vector of G corresponding to $\rho_\alpha(G)$. By the α eigenequation of G at vertex u we have

$$\rho_\alpha x_u = \alpha T_u x_u + (1-\alpha)\sum_{u,v\in V(G)} d_{uv}x_v.$$

Let $x_u = \max_{1\leqslant j\leqslant n}\{x_j\}$

$$\rho_\alpha x_u = \alpha T_u x_u + (1-\alpha)\sum_{v=1}^{n} d_{uv}x_v,$$

$$\rho_\alpha x_u \leqslant \alpha T_u x_u + (1-\alpha)\sum_{v=1}^{n} d_{uv}x_u,$$

$$\rho_\alpha x_u \leqslant T_u x_u. \tag{3.3}$$

Now to get the quadratic form we have,

$$\rho_\alpha^2 x = D_\alpha^2 x = (\alpha T(G) + (1-\alpha)D)^2 x,$$

$$= \alpha^2 T_G^2 x + (1+\alpha^2-2\alpha)D^2x + \alpha(1-\alpha)T_G Dx + \alpha(1-\alpha)DT_G x.$$

Thus,

$$\rho_\alpha^2 x_u = \alpha^2 T_u^2 x_u + (1+\alpha^2-2\alpha) \times \sum_{v=1}^{n} d_{uv}\sum_{w=1}^{n} d_{vw}x_w + \alpha(1-\alpha)T_u\sum_{v=1}^{n} d_{uv}x_v + \alpha(1-\alpha)\sum_{v=1}^{n} d_{uv}T_v x_v.$$

In order to prove (5) we consider the simple quadratic function of $\rho_\alpha(G)$,

$$(\rho_\alpha^2 + b\rho_\alpha)x_u = \alpha^2 T_u^2 x_u + (1+\alpha^2-2\alpha) \times \sum_{v=1}^{n} d_{uv} \sum_{w=1}^{n} d_{vw}x_w + \alpha(1-\alpha)(T_u \sum_{v=1}^{n} d_{uv}x_v + \sum_{v=1}^{n} d_{uv}T_v x_v) + b\rho_\alpha x_u,$$

where b is an integer. As we know $x_u = \max_{1\leqslant j\leqslant n}\{x_j\}$,

$$\sum_{v=1}^{n} d_{uv}T_v x_v \leqslant T_u m_u x_u; \quad \sum_{v=1}^{n} d_{uv} \sum_{w=1}^{n} d_{vw}x_w \leqslant T_u m_u x_u; \quad \sum_{v=1}^{n} d_{uv}x_v \leqslant T_u x_u,$$

by the above inequalities we get a simple quadratic equation, provided that $\alpha T_u + b \geqslant 0$ we get,

$$(\rho_\alpha^2 + b\rho_\alpha)x_u \leqslant (\alpha T_u + b)T_u x_u + (1-\alpha)T_u m_u x_u,$$
$$\rho_\alpha^2 + b\rho_\alpha - ((\alpha T_u + b + (1-\alpha)m_u)T_u \leqslant 0.$$

Since ρ_α is the Perron vector and $\rho_\alpha \geqslant 0$,

$$\rho_\alpha(G) \leqslant \frac{-b + \sqrt{b^2 + 4T_u(\alpha T_u + b + (1-\alpha)m_u)}}{2}.$$

From the inequality above for each b we can get a distinct upper bound. In particular if $b = -\alpha T_u$ then we get the inequality (5).

$$\rho_\alpha(G) \leqslant \max_{1\leqslant u\leqslant n} \frac{\alpha T_u + \sqrt{4T_u m_u(1-\alpha) + (\alpha T_u)^2}}{2}.$$

Suppose the equality holds in above inequality. Then all inequalities in the above arguments must be equalities. Therefore we have $x_u = x_v$ for all v. By $\rho_\alpha x = D_\alpha x$ we can deduce that $T_1 = T_2 = \cdots = T_n$, that is, G is a transmission regular graph. □

Theorem 3.4. *Let G be a simple connected graph with n vertices and transmission sequence $(T_1, \ldots, T_n)$ and $\alpha \in [1/2, 1]$. Then*

$$\rho_\alpha(G) \leqslant \max_{1\leqslant u\neq w\leqslant n} \frac{\beta + \sqrt{4(1-\alpha)d_{uw}T_u + \beta^2}}{2}, \tag{3.4}$$

where $\beta = T_w - (1-\alpha)d_{uw}$. Moreover, if G is the transmission regular graph then equality holds.

Proof. Let $x = (x_1, \ldots, x_n)^T$ be the distance α-Perron vector of G corresponding to $\rho_\alpha(G)$. We can assume that one eigencomponent x_u is equal to 1 and other components are less than or equal to 1 i.e., $x_u = 1$ and $0 \leqslant x_v \leqslant 1$ for all x_v. Let $x_w = \max\{x_v | v \neq u\}$. So by the D_α-eigenequation:

$$\rho_\alpha(G)x_u = \alpha T_u x_u + (1-\alpha)\sum_{v=1}^{n} d_{uv}x_v,$$

$$\rho_\alpha(G)x_u \leqslant \alpha T_u x_u + (1-\alpha)T_u x_w. \tag{3.5}$$

Now by using the α eigenequation corresponding to x_w,

$$\rho_\alpha(G)x_w = \alpha T_w x_w + (1-\alpha)\sum_{v=1}^{n} d_{wv}x_v.$$

By expanding the above equation,

$$\rho_\alpha(G)x_w = \alpha T_w x_w + (1-\alpha)d_{wu}x_u + (1-\alpha)(d_{wu}x_w + \sum_{v\neq u} d_{wv}x_v) - (1-\alpha)d_{uw}x_w,$$

$$\rho_\alpha(G)x_w \leqslant T_w x_w + (1-\alpha)d_{uw}x_u - (1-\alpha)d_{uw}(x_w). \tag{3.6}$$

Now multiplying both sides of equation (3.6) with $\rho_\alpha(G)$ we have

$$\rho_\alpha^2(G)x_w \leqslant T_w\rho_\alpha(G)x_w + (1-\alpha)d_{uw}\rho_\alpha(G)x_u - (1-\alpha)d_{uw}\rho_\alpha(G)x_w.$$

Substituting equation (3.5) i.e., $\rho_\alpha(G)x_u \leqslant \alpha T_u x_u + (1-\alpha)T_u x_w$ in the above inequality we get,

$$\begin{aligned}\rho_\alpha^2(G)x_w \leqslant & T_w\rho_\alpha(G)x_w + (1-\alpha)d_{uw}\big(\alpha T_u x_u + (1-\alpha)T_u x_w\big)\\ & - (1-\alpha)d_{uw}\rho_\alpha(G)x_w.\end{aligned}$$

$$\left(\rho_\alpha^2(G) - T_w\rho_\alpha(G) + (1-\alpha)d_{uw}\rho_\alpha(G) - (1-\alpha)^2 T_u d_{uw}\right)x_w \leqslant \alpha(1-\alpha)d_{uw}T_u x_u.$$

As we already know that $x_u = 1$ and $x_w = \max\{x_v | v \neq u\}$ for all x_v. Thus,

$$\left(\rho_\alpha^2(G) - T_w\rho_\alpha(G) + (1-\alpha)d_{uw}\rho_\alpha(G) - (1-\alpha)^2 T_u d_{uw}\right) \leqslant \alpha(1-\alpha)d_{uw}T_u.$$

So we get the quadratic form as,

$$\rho_\alpha^2(G) - \left(T_w - (1-\alpha)d_{uw}\right)\rho_\alpha(G) - (1-\alpha)T_u d_{uw} \leqslant 0.$$

Thus we get the inequality (3.4).

$$\rho_\alpha(G) \leqslant \max_{1\leqslant u\neq w\leqslant n} \frac{\beta + \sqrt{4(1-\alpha)d_{uw}T_u + \beta^2}}{2},$$

where $\beta = T_w - (1-\alpha)d_{uw}$.

Suppose the equality occurs in Equation 3.4, then x is an eigenvector of $\rho_\alpha(G)$ with $x_u = x_w$ for all $w = 1, 2, \ldots, n$. This concludes that all the row sums of $D_\alpha(G)$ are same, and so G is a transmission regular graph. □

Theorem 3.5. *Let G be a simple connected graph on n vertices with transmission sequence $(T_1, \ldots, T_n)$, where d is the diameter of G and $\alpha \in [1/2, 1]$. Then*

$$\rho_\alpha \leqslant \max_{1\leqslant u\leqslant n} \frac{\alpha T_u + \sqrt{(\alpha T_u)^2 + \dfrac{4n(1-\alpha)d}{T_u}\displaystyle\sum_{v\neq u} T_v(\alpha T_v + (1-\alpha)m_v)}}{2}, \tag{3.7}$$

where $m_v = (1/T_v)\sum_{w=1}^{n} d_{wv}T_w$. *Moreover, the equality holds if and only if* G *is a complete graph.*

Proof. Let $B = (b_1, \ldots, b_n)$ is an $n \times n$ diagonal matrix where $b_u > 0$. Consider the matrix $B^{-1}D_\alpha B$. Since D_α and $B^{-1}D_\alpha B$ are similar matrices, $\rho_\alpha(G)$ is also an eigenvalue of $B^{-1}D_\alpha B$. We see that $(u, v)^{th}$-entry of $B^{-1}D_\alpha B$ is αT_u for $u = v$ or $(1-\alpha)(b_v/b_u)d_{uv}$ for $u \neq v$.

Let $x = (x_1, \ldots, x_n)^T$ be the distance α-Perron vector of G corresponding to $\rho_\alpha(G)$. We can assume that one eigencomponent x_u is equal to 1 and other components are less than or equal to 1 i.e., $x_u = 1$ and $0 \leqslant x_v \leqslant 1$ for all x_v then by the α eigenequation for x_u,

$$\rho_\alpha x_u = \alpha T_u x_u + (1-\alpha)\sum_{v=1}^{n} \frac{b_v d_{uv}}{b_u} x_v,$$

$$\rho_\alpha = \alpha T_u + (1-\alpha)\sum_{v=1}^{n} \frac{b_v d_{uv}}{b_u} x_v.$$

Now by the α eigenequation for x_v:

$$\rho_\alpha x_v = \alpha T_v x_v + (1-\alpha)\sum_{w=1}^{n} \frac{b_w d_{vw}}{b_v} x_w,$$

$$\rho_\alpha^2 = \alpha\rho_\alpha T_u + (1-\alpha)\sum_{v=1}^{n} \frac{b_v d_{uv}}{b_u} \rho_\alpha x_v,$$

$$\rho_\alpha^2 = \alpha\rho_\alpha T_u + \frac{\alpha(1-\alpha)}{b_u}\sum_{v=1}^{n} b_v d_{uv} T_v x_v + \frac{(1-\alpha)^2}{b_u}\sum_{v=1}^{n} d_{uv}\sum_{w=1}^{n} b_w d_{vw} x_w.$$

On other hand d is the diameter of G and $0 \leqslant x_v \leqslant 1$, so $nd \geqslant \sum_{v=1}^{n} d_{uv}x_v$. Thus,

$$\rho_\alpha^2 \leqslant \alpha\rho_\alpha T_u + \frac{\alpha(1-\alpha)nd}{b_u}\sum_{v\neq u} b_v T_v + \frac{(1-\alpha)^2 nd}{b_u}\sum_{v\neq u}\sum_{w=1}^{n} b_w d_{vw}.$$

Moreover, by setting up $b_w = T_w$ for all w, we have

$$\rho_\alpha^2 \leqslant \alpha\rho_\alpha T_u + \frac{(1-\alpha)nd}{T_u}\sum_{v\neq u} T_v(\alpha T_v + (1-\alpha)m_v),$$

where $m_v = (1/T_v)\sum_{w=1}^{n} d_{wv}T_w$. Hence we obtain the inequality (3.7).

$$\rho_\alpha \leqslant \max_{1\leqslant u\leqslant n} \frac{\alpha T_u + \sqrt{(\alpha T_u)^2 + \frac{4n(1-\alpha)d}{T_u}\sum\limits_{v\neq u} T_v(\alpha T_v + (1-\alpha)m_v)}}{2}.$$

Suppose the equality hold is the above equation, then all inequalities in the above argument must be equalities. In particular $d_{uv} = d$, then $d = 1$ since G is connected. Hence G must be the complete graph K_n. $\square$

Theorem 3.6. *Let G be a simple connected graph with transmission sequence $(T_1,\dots,T_n)$ and $\alpha \in [1/2,1]$. Then*

$$\rho_\alpha(G) \geqslant \frac{1}{\sum\limits_{i=1}^{n} T_i^2} \sum_{u,v\in V(G)} d_{uv}(\alpha(T_u-T_v)^2+2T_uT_v). \tag{3.8}$$

Moreover, the equality holds if G is the transmission regular graph.

Proof. Let

$$x = \frac{1}{\sqrt{\sum\limits_{i=1}^{n} T_i^2}}(T_1,\dots,T_n)^T$$

be a unit vector, then by using Rayleigh Quotient,

$$\rho_\alpha(G) \geqslant x^T D_\alpha x,$$

from Equation (2.2) we have,

$$x^T D_\alpha(G)x = \frac{1}{\sum\limits_{i=1}^{n} T_i^2} \sum_{\{u,v\}\in V(G)} (\alpha(T_u^2+T_v^2)+2(1-\alpha)T_uT_v)$$

$$x^T D_\alpha(G)x = \frac{1}{\sum\limits_{i=1}^{n} T_i^2} \sum_{u,v\in V(G)} d_{uv}(\alpha(T_u-T_v)^2+2T_uT_v).$$

So we have,

$$\rho_\alpha(G) \geqslant \frac{1}{\sum\limits_{i=1}^{n} T_i^2} \sum_{u,v\in V(G)} d_{uv}(\alpha(T_u-T_v)^2+2T_uT_v).$$

Hence we obtain the bound (3.8). Moreover, it is easy to check that the equality holds if G is a transmission regular graph. $\square$

Theorem 3.7. *Let G be a simple connected graph with n vertices and transmission sequence $(T_1,\dots,T_n)$ and $\alpha \in [1/2,1]$, then*

$$\rho_\alpha(G) \geqslant \min_{1\leqslant i\neq j\leqslant n} \frac{\gamma_{ij}+\sqrt{\gamma_{ij}^2+4T_i(d_{ij}-\alpha(T_j+d_{ij}))}}{2}, \tag{3.9}$$

where $\gamma_{ij} = \alpha(T_i+d_{ij})+T_j-d_{ij}$. Moreover, if G is a transmission regular graph then equality holds.

Proof. Let $x=(x_1,\dots,x_n)^T$ be the eigenvector corresponding to eigenvalue $\rho_\alpha(G)$ of $D_\alpha(G)$. Then by the eigenequation,

$$D_\alpha(G)x = \rho_\alpha(G)x.$$

From the k^{th} equation of above equation we have,

$$\rho_\alpha(G)x_k = \alpha T_k x_k + (1-\alpha)\sum_{v_l\in V(G)} d_{kl}x_k, \qquad k=1,2,\dots,n.$$

Since $D_\alpha(G)$ is irreducible and nonnegative, so we have $x_k > 0$ for all $k = 1, 2, \dots, n$. We can assume that x_i and x_j are the minimum and second minimum components of eigenvector x respectively i.e.,

$$x_i = \min_{v_k \in V(G)} x_k, \qquad x_j = \min_{v_k \in V(G), k \neq i} x_k,$$

$$x_k \geqslant x_j \geqslant x_i > 0.$$

It is obvious that for $v_k \in V(G)$, where $v_k \neq i, j$. For $v_i \in V(G)$ using the eigenequation,

$$\rho_\alpha(G) x_i = \alpha T_i x_i + (1-\alpha) \sum_{v_k \in V(G)} d_{ik} x_k,$$

$$\geqslant \alpha T_i x_i + (1-\alpha) T_i x_j,$$

$$(\rho_\alpha(G) - \alpha T_i) x_i \geqslant (1-\alpha) T_i x_j. \tag{3.10}$$

Thus by using an eigenequation for $v_j \in V(G)$,

$$\rho_\alpha(G) x_j = \alpha T_j x_j + (1-\alpha) \sum_{v_k \in V(G)} d_{jk} x_k,$$

$$\geqslant \alpha T_j x_j + (1-\alpha) d_{ij} x_i + (1-\alpha)(T_j - d_{ij}) x_j,$$

$$(\rho_\alpha(G) - T_j + d_{ij} - \alpha d_{ij}) x_j \geqslant (1-\alpha) d_{ij} x_i. \tag{3.11}$$

From equations (3.10) and (3.11) we see that left hand side of inequalities are positive so we can multiply both inequalities as,

$$(\rho_\alpha(G) - \alpha T_i)(\rho_\alpha(G) - T_j + d_{ij} - \alpha d_{ij}) \geqslant (1-\alpha)^2 d_{ij} T_i.$$

Now by solving the above inequality we get the quadratic form,

$$\rho_\alpha^2(G) - (\alpha(d_{ij} + T_i) + T_j - d_{ij}) \rho_\alpha(G) + (\alpha(T_j + d_{ij}) - d_{ij}) T_i \geqslant 0$$

that is,

$$\rho_\alpha(G) \geqslant \frac{\gamma_{ij} + \sqrt{\gamma_{ij}^2 + 4T_i(d_{ij} - \alpha(T_j + d_{ij}))}}{2},$$

where $\gamma_{ij} = \alpha(T_i + d_{ij}) + T_j - d_{ij}$. From the above inequality we get the required result in (3.9).

Suppose the equality holds in above inequality. Then all inequalities in the above arguments must be equalities. Therefore we have $x_i = x_j$ for all $j = 1, 2, \dots, n$. By $\rho_\alpha(G) x = D_\alpha(G) x$ we can deduce that $T_1 = T_2 = \cdots = T_n$, that is, G is a transmission regular graph. □

A subset S of a vertex set $V(G)$ of a graph G is said to be an *independent set* if no two vertices of S are adjacent in G. The independence number of G is the maximum number of vertices in the independent sets in G. The following theorems will give the lower bound for D_α spectral radius in terms of the order and the independence number of G.

Theorem 3.8. *Let G be a connected graph of order n with independence number s and $\alpha \in [1/2, 1]$. Then*

$$\rho_\alpha(G) \geqslant \frac{\theta - \sqrt{\theta^2 - 4(ns(n\alpha^2 - 4\alpha + 1) - s^2(n\alpha^2 - 2n\alpha + 1) + 2s\alpha)}}{2},$$

where $\theta = -(2s + ns\alpha - 2s\alpha + n\alpha - 2)$.

Proof. Let G be a connected graph of order n and S be the independent set with independence number s. Let $X = (x_1, \ldots, x_n)^T$ be the Perron vector corresponding to $\rho_\alpha(G)$, where x_i corresponds to vertex v_i for $i = 1, 2, \ldots, n$. As S is independent set so $u, v \in V(S)$, $d(u, v) \geqslant 2$ as all the vertices are nonadjacent in S. Assume that $x_i = \min\{k \mid k \in V(S)\}$ and $x_j = \min\{k \mid k \in V(G)\backslash S\}$.

By using $\rho_\alpha(G)$ eigenequation for x_i we have,

$$\rho_\alpha(G)x_i = \sum_{v \in V(S)} d_{iv}(\alpha x_i + (1-\alpha)x_v) + \sum_{u \in V(G)\backslash S} d_{iu}(\alpha x_i + (1-\alpha)x_u).$$

As x_i , x_j are the minimum eigenvector components of $V(S)$ and $V(G)\backslash S$ respectively. So it follows,

$$\rho_\alpha(G)x_i \geqslant 2(s-1)(\alpha x_i + (1-\alpha)x_v) + (n-s)(\alpha x_i + (1-\alpha)x_v),$$

$$\rho_\alpha(G)x_i \geqslant (2(s-1) + \alpha(n-s))x_i + (1-\alpha)(n-s)x_j.$$

As left hand side of the above inequality is positive so we have,

$$\Big(\rho_\alpha(G) - (2(s-1) + \alpha(n-s))\Big)x_i \geqslant (1-\alpha)(n-s)x_j. \tag{3.12}$$

Similarly using $\rho_\alpha(G)$ eigenequation for x_j we have,

$$\rho_\alpha(G)x_j = \sum_{v \in V(S)} d_{jv}(\alpha x_j + (1-\alpha)x_v) + \sum_{u \in V(G)\backslash S} d_{ju}(\alpha x_j + (1-\alpha)x_u).$$

As x_i , x_j are the minimum eigenvector components of $V(S)$ and $V(G)\backslash S$ respectively. So it follows,

$$\rho_\alpha(G)x_j \geqslant s(1-\alpha)x_i + \alpha s(n-1)x_j.$$

As the left hand side of the above inequality is positive so we have,

$$\Big(\rho_\alpha(G) - \alpha s(n-1)\Big)x_j \geqslant s(1-\alpha)x_i. \tag{3.13}$$

Thus multiplying inequalities (3.12) and (3.13) we get the quadratic form as:

$$\rho_\alpha^2(G) - \Big(2s + ns\alpha - 2s\alpha + n\alpha - 2\Big)\rho_\alpha(G)$$
$$+ ns(n\alpha^2 - 4\alpha + 1) - s^2(n\alpha^2 - 2n\alpha + 1) + 2s\alpha \geqslant 0.$$

So we get the required bound. □

Theorem 3.9. *Let $G = (V, E)$ be a connected bipartite graph of order n with bipartition $V(G) = P \cup Q$ where $|P| = p$ and $|Q| = q$ and $\alpha \in [1/2, 1]$. Then*

$$\rho_\alpha(G) \geqslant \frac{(2+\alpha)(n) - 4 + \sqrt{(4 - 6\alpha + \alpha^2)(p^2+q^2) + pq(\alpha - 1 - 6\alpha^2)}}{2}, \tag{3.14}$$

$$\rho_\alpha(G) \leqslant \frac{(n-1)\Big((\alpha n + n - 2) + \sqrt{(\alpha n + n - 2)^2 - 4pq(1-\alpha)^2}\Big)}{2}. \tag{3.15}$$

Moreover, if G is a complete bipartite graph then equality holds in (3.14) and if G is a path of order $p + q$ then equality holds in (3.15).

Proof. Since G is a bipartite graph with bipartition $V(G) = P \cup Q$, where $|P| = p$ and $|Q| = q$. As $P \cap Q = \emptyset$ so we assume that $P = \{1, 2, \ldots, p\}$ and $Q = \{p+1, p+2, \ldots, p+q\}$ with $p + q = n$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the eigenvector of $D_\alpha(G)$ corresponding to the eigenvalue $\rho_\alpha(G)$. We can assume that $x_i = \min\{x_k | k \in P\}$ and also $x_j = \min\{x_k | k \in Q\}$. So for $i \in P$ using the eigenequation,

$$\rho_\alpha(G)x_i = \sum_{v=1, v\neq i}^{p} d_{iv}(\alpha x_i + (1-\alpha)x_v) + \sum_{u=p+1}^{p+q} d_{iu}(\alpha x_i + (1-\alpha)x_u).$$

As x_i and x_j are the minimums in P and Q respectively, so we have

$$\rho_\alpha(G)x_i \geqslant (2(p-1) + \alpha q)x_i + ((1-\alpha)q)x_j.$$

We see that the left hand side of the above inequality is positive. Thus

$$\Big(\rho_\alpha(G) - (2(p-1) + \alpha q)\Big)x_i \geqslant ((1-\alpha)q)x_j. \tag{3.16}$$

Similarly for $j \in Q$ using the eigenequation we have,

$$\rho_\alpha(G)x_j = \sum_{v=1}^{p} d_{jv}(\alpha x_j + (1-\alpha)x_v) + \sum_{u=p+1, u\neq j}^{p+q} d_{ju}(\alpha x_j + (1-\alpha)x_u)$$

$$\rho_\alpha(G)x_j \geqslant (2(q-1) + \alpha p)x_j + ((1-\alpha)p)x_i.$$

We see that the left hand side of the above inequality is positive. Thus

$$\Big(\rho_\alpha(G) - (2(q-1) + \alpha p)\Big)x_j \geqslant ((1-\alpha)p)x_i. \tag{3.17}$$

Since G is connected, so $x_k > 0$ for all $k \in V(G)$. Multiplying the inequalities (3.16) and (3.17) we have,

$$\begin{aligned}&\rho_\alpha^2(G) + (4 - (p+q)(2+\alpha))\rho_\alpha(G) + 2\alpha(p^2+q^2)\\ &- (p+q)(2\alpha+4) + pq(5 + 2\alpha^2 - 2\alpha) + 4 \geqslant 0.\end{aligned}$$

From above quadratic inequality we have,

$$\rho_\alpha(G) \geqslant \frac{(2+\alpha)(n) - 4 + \sqrt{(4-6\alpha+\alpha^2)(p^2+q^2) + pq(\alpha-1-6\alpha^2)}}{2}.$$

Now suppose that equality holds. Then all inequalities in the above argument must be equalities. From equality in (3.16), we get $x_k = x_j$ for all $k \in V(Q)$. From equality in (3.17), we get $x_k = x_i$ for all $k \in V(P)$. Thus, each vertex in each set is adjacent to all the vertices on the other set and vice versa. Hence, G is a complete bipartite graph which completes the proof for (3.14). Now for the upper bound in equation (3.15) we can assume that $x_i = \max\{x_k | k \in P\}$ and also $x_j = \max\{x_k | k \in Q\}$. So for $i \in P$ using the eigenequation,

$$\rho_\alpha(G)x_i = \sum_{v=1, v\neq i}^{p} d_{iv}(\alpha x_i + (1-\alpha)x_v) + \sum_{u=p+1}^{p+q} d_{iu}(\alpha x_i + (1-\alpha)x_u)$$

$$\Big(\rho_\alpha(G) - (p-1+\alpha q)(p+q-1)\Big)x_i \leqslant q(1-\alpha)(p+q-1)x_j. \tag{3.18}$$

Similarly for $j \in Q$ using the eigenequation we have,

$$\rho_\alpha(G)x_j = \sum_{v=1}^{p} d_{jv}(\alpha x_j + (1-\alpha)x_v) + \sum_{u=p+1, u\neq j}^{p+q} d_{ju}(\alpha x_j + (1-\alpha)x_u)$$

$$\Big(\rho_\alpha(G) - (\alpha p + q - 1)(p+q-1)\Big)x_j \leqslant p(1-\alpha)(p+q-1)x_i. \tag{3.19}$$

Since G is connected, so $x_k > 0$ for all $k \in V(G)$. Multiplying the inequalities (3.18) and (3.19) we have,

$$\rho_\alpha^2(G) - (n-1)(\alpha n + n - 2)\rho_\alpha(G) + pq(n-1)^2(1-\alpha)^2) \leqslant 0.$$

From the above quadratic inequality we get

$$\rho_\alpha(G) \leqslant \frac{(n-1)\Big((\alpha n + n - 2) + \sqrt{(\alpha n + n - 2)^2 - 4pq(1-\alpha)^2}\Big)}{2}.$$

From the above inequality, it can easily be verified that equality holds if G is a path graph of order $p+q$. □

Graph operations are natural techniques for producing new graphs from old ones. The *join of two vertex disjoint connected graphs* G and H, denoted by $G \vee H$ is the graph obtained from the union $G \cup H$ by joining each vertex of G to each vertex of H.

Theorem 3.10. *The graph $K_{1,n-1}$ is a unique graph which maximizes the D_α spectral radius among all graphs with diameter 2.*

Proof. Let G be a connected graph of diameter 2 with $V(G) = \{v_1, v_2, \dots, v_n\}$. Suppose $X = (x_1, x_2, \dots, x_n)^T$ is a Perron vector of G corresponding to $\rho_\alpha(G)$, where each x_i corresponds to vertex v_i, $(i = 1, 2, \dots, n)$. Let $v_t \in V(G)$ such that $x_t = \min\{x_i | v_i \in V(G)\}$. Let $d(v_t)$ denotes the degree of vertex v_t. Then we consider the following two cases.

CASE 1: $d(v_t) = n - 1$.

If we delete all the edges in $N(v_t)$, then the resulting graph will be $K_{1,n-1}$. Hence from Lemma 2.1 we have $\rho_\alpha(K_{1,n-1}) \geqslant \rho_\alpha(G)$ and equality holds if $G \cong K_{1,n-1}$.

CASE 2: $d(v_t) \leqslant n - 2$.

Let $C(v_t)$ be the set of vertices in G that are not adjacent to v_t. Then $C(v_t) \neq \emptyset$, obviously each vertex in $C(v_t)$ is adjacent to atleast one vertex in $N(v_t)$. As we know from Equation (2.2) that for any vertex u and v in G,

$$x^T D_\alpha(G) x = \sum_{\{u,v\} \in V(G)} d_{uv}(\alpha(x_u^2 + x_v^2) + 2(1-\alpha)x_u x_v).$$

Now let G^* be the graph obtained from G such that

$$G^* = G - [N(v_t), C(v_t)] + \{v_t v_i | v_i \in C(v_t)\},$$

clearly the diameter of G^* is 2 and maximum degree i.e., $\Delta(G^*) = n-1$. As we move from G to G^*, we see that the distance between v_t and $C(v_t)$ is decreased by 1, while the distance between $N(v_t)$ and $C(v_t)$ is increased by 1. Using the Rayleigh Quotient we have,

$$\rho_\alpha(G^*) - \rho_\alpha(G) \geqslant x^T(D(G^*) - (D(G)))x,$$

$$\begin{aligned}
\rho_\alpha(G^*) - \rho_\alpha(G) \geqslant & \sum_{v \in C(v_t)} (\alpha(x_t^2 + x_v^2) + 2(1-\alpha)x_t x_v) \\
& + 2 \sum_{\substack{v_i, v_j \in \\ [N(v_t), C(v_t)]}} (\alpha(x_i^2 + x_j^2) + 2(1-\alpha)x_i x_j) \\
& - 2 \sum_{v \in C(v_t)} (\alpha(x_t^2 + x_v^2) + 2(1-\alpha)x_t x_v) \\
& - \sum_{\substack{v_i, v_j \in \\ [N(v_t), C(v_t)]}} (\alpha(x_i^2 + x_j^2) + 2(1-\alpha)x_i x_j)
\end{aligned}$$

$$\begin{aligned}
\rho_\alpha(G^*) - \rho_\alpha(G) \geqslant & \sum_{\substack{v_i, v_j \in \\ [N(v_t), C(v_t)]}} (\alpha(x_i^2 + x_j^2) + 2(1-\alpha)x_i x_j) \\
& - \sum_{v \in C(v_t)} (\alpha(x_t^2 + x_v^2) + 2(1-\alpha)x_t x_v).
\end{aligned}$$

As we assumed $x_t = \min\{x_i | v_i \in V(G)\}$, therefore we have $\rho_\alpha(G^*) \geqslant \rho_\alpha(G)$. By Case 1 we see that $\rho_\alpha(K_{1,n-1}) \geqslant \rho_\alpha(G^*) \geqslant \rho_\alpha(G)$.

If $\rho_\alpha(K_{1,n-1}) = \rho_\alpha(G)$, then $G^* \cong K_{1,n-1}$ and $\rho_\alpha(G^*) = \rho_\alpha(G)$. It follows that X is also the perron vector of $K_{1,n-1}$. Then if $v_i, v_j \neq v_t$, then we have $x_i = x_j > x_t$. Since all the above inequalities are equalities so we have

$$\sum_{\substack{v_iv_j\in\\ [N(v_t),C(v_t)]}} (\alpha(x_i^2+x_j^2)+2(1-\alpha)x_ix_j) = \sum_{v_i\in C(v_t)} (\alpha(x_t^2+x_v^2)+2(1-\alpha)x_tx_v).$$

Then for each edge $v_iv_j \in [N(v_t), C(v_t)], v_i \in N(v_t)$ and $v_j \in C(v_t)$, we get $x_i = x_t$, a contradiction. Hence $\rho_\alpha(K_{1,n-1}) > \rho_\alpha(G)$.

□

Remark: It can be easily seen from Lemmas 2.2 and 2.3 that $K_n - e$ is unique graph that minimizes the D_α spectral radius among all graphs of diameter 2.

The *complement* of a graph G denoted by $\bar{G}$ is a graph on the same vertices such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G. In the following result, we show the lower bound for the sum of D_α-spectral radius of G and $\bar{G}$.

Theorem 3.12. *Let G be a connected graph of order $n \geqslant 4$ vertices and m edges. Let $\bar{G}$ is also connected. Then*

$$\rho_\alpha(G) + \rho_\alpha\bar{G} \geqslant 3(n-1).$$

Proof. Let G and $\bar{G}$ be connected. Let $v \in V(G) = V(\bar{G})$ and δ_v, $\bar{\delta_v}$ be the degrees of v in G and $\bar{G}$ respectively. Then

$$T_v \geqslant \delta_v + 2(n-1-\delta_v) = 2(n-1) - \delta_v,$$

where equality holds if the maximum distance between v and any other vertex of G is 2. Let m and $\bar{m}$ be the number of edges in G and $\bar{G}$ respectively. Then

$$\xi(G) = \frac{1}{2}\sum_{v\in V(G)} T_v \geqslant \frac{1}{2}\sum_{v\in V(G)} [2(n-1)-\delta_v] = n(n-1) - m,$$

where equality holds if and only if G is of diameter 2. Similarly

$$\xi(\bar{G}) = \frac{1}{2}\sum_{v\in V(\bar{G})} T_v \geqslant \frac{1}{2}\sum_{v\in V(\bar{G})} [2(n-1)-\bar{\delta_v}] = n(n-1) - \bar{m},$$

where equality holds if the maximum distance between v and any other vertex of G is 2. Now note that $m + \bar{m} = (1/2)n(n-1)$.

By using Lemma 3.1 we have,

$$\rho_\alpha(G) \geqslant \frac{2\xi(G)}{n},$$

$$\rho_\alpha(G) + \rho_\alpha(\bar{G}) \geqslant \frac{2(\xi(G)+\xi(\bar{G}))}{n} \geqslant \frac{2}{n}[2n(n-1) - (m+\bar{m})],$$

$$\rho_\alpha(G) + \rho_\alpha(\bar{G}) \geqslant 3(n-1).$$

The equalities in the above inequalities hold if and only if G and $\bar{G}$ are both transmission regular and diameter is at most two. Since $n \geqslant 4$, if G ($\bar{G}$, respectively) is of diameter one then $G = K_n$ ($\bar{G} = K_n$, respectively), which contradicts the fact that G and $\bar{G}$ is connected. The result follows since $T_v(G) = 2(n-1) - \delta_v$ and $T_v(\bar{G}) = 2(n-1) - \bar{\delta_v}$ for each $v \in V(G) = V(\bar{G})$, if G and $\bar{G}$ are both of diameter two. $\square$

4. Discussion

Let G_i be connected graphs for $i = 1, 2, 3, 4$ in Figure 2. In this section of the paper, we briefly discuss lower and upper bounds for D_α spectral radius by showing examples for each bound for G_i (see Figure 2) and then comparing them with the exact D_α-spectral radius. We see that $G_4 = C_6$ is a transmission regular graph and each vertex has transmission 9.

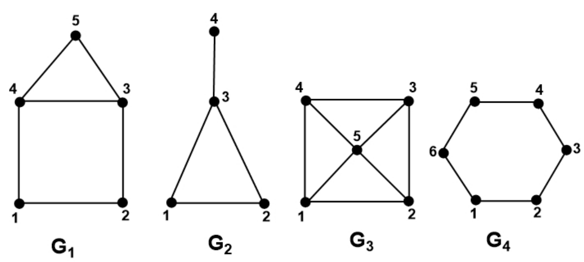

Figure 2. An example of simple connected graphs.

In Table 1, we depict the upper bounds up to four decimal places for Theorems 3.3, 3.4 and 3.5 for $\rho_\alpha(G_i)$ for fixed $\alpha = 3/4$.

	G_1	G_2	G_3	G_4
$\rho_\alpha(G)$	5.7219	4.3314	4.8829	9
Theorem 3.3	5.8986	4.7500	4.9597	9
Theorem 3.4	6	5.0012	5.0010	9
Theorem 3.5	12.1625	6.9433	8.8850	19.9832

Table 1. Table of upper bounds for $\rho_\alpha(G_i)$ for $i = 1, 2, 3, 4$.

In Table 2, we depict the lower bounds up to four decimal places for Theorems 3.6, 3.7 and 3.8 for $\rho_\alpha(G_i)$ for fixed $\alpha = 3/4$.

	G_1	G_2	G_3	G_4
$\rho_\alpha(G)$	5.7219	4.3314	4.8829	9
Theorem 3.6	5.6709	4.2045	4.8530	9
Theorem 3.7	5	3.8661	4.8727	9
Theorem 3.8	4.5001	4	4.5100	3.3590

TABLE 2. Table of lower bounds for $\rho_\alpha(G_i)$ for $i = 1, 2, 3, 4$.

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