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STARRED ITALIAN DOMINATION IN GRAPHS

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Abstract. An Italian dominating function on a graph G is a function $f : V(G) \to \{0,1,2\}$ such that $\sum_{u\in N(v)} f(u) \geq 2$ for every vertex $v \in V_0$, where $V_0 = \{v \in V(G) : f(v) = 0\}$ and $N(v)$ represents the open neighbourhood of v. A starred Italian dominating function on G is an Italian dominating function f such that V_0 is not a dominating set of G. The starred Italian domination number of G, denoted $\gamma_I^*(G)$, is the minimum weight $\omega(f) = \sum_{v\in V(G)} f(v)$ among all starred Italian dominating functions f on G. In this article, we initiate the study of the starred Italian domination in graphs. For instance, we give some relationships that exist between this parameter and other domination invariants in graphs. Also, we present tight bounds and characterize the extreme cases. In addition, we obtain exact formulas for some particular families of graphs. Finally, we show that the problem of computing the starred Italian domination number of a graph is NP-hard.

1. Introduction

Throughout this article we only consider simple graphs G with vertex set $V(G)$ and edge set $E(G)$. Given a vertex v of G, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the *open neighbourhood* and the *closed neighbourhood* of v in G, respectively. A set $D \subseteq V(G)$ of vertices is a *dominating set* of G if every vertex $v \in V(G) \setminus D$ has at least one neighbour in D, that is, $|N(v) \cap D| \geq 1$. The *domination number* of G is the minimum cardinality among all dominating sets of G and it is denoted by $\gamma(G)$.

In the last decades, dominating sets and their variants in graphs have been interesting topics in graph theory. For instance, in the books [11, 12, 17] the authors expose some varieties of dominating sets D, which depend on conditions that can be imposed either on the set D, or on the set $V(G) \setminus D$, or on the "method" by which vertices in $V(G) \setminus D$ are dominated. We next remark on two of these variations.

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Multiple domination. This concept was introduced by Fink and Jacobson [7]. A dominating set D of a graph G is a *k-dominating set* if every vertex in $V(G) \setminus D$ has at least k neighbour vertices in D, i.e., $|N(v) \cap D| \geq k$ for every vertex v in $V(G) \setminus D$. The *k-domination number* of G is the minimum cardinality among all k-dominating sets of G and it is denoted by $\gamma_k(G)$.
Maximal domination. This parameter was introduced by Kulli and Janakiram [16]. A dominating set D of a graph G is a *maximal dominating set* if $V(G) \setminus D$ is not a dominating set of G. The *maximal domination number* of G is the minimum cardinality among all maximal dominating sets of G and it is denoted by $\gamma_m(G)$.

Dominating functions in domination theory have been extensively studied. One of the reasons may be due to the fact that dominating functions generalize the concept of dominating sets. For an arbitrary subset $\mathcal{P}$ of the reals, a function $f : V(G) \to \mathcal{P}$ is said to be a *$\mathcal{P}$-dominating function* on a graph G if $f(N[v]) \geq 1$ for every $v \in V(G)$. The *weight* of a $\mathcal{P}$-dominating function f on a set $S \subseteq V(G)$ is $f(S) = \sum_{v \in S} f(v)$. If particularly $S = V(G)$, then $f(V(G))$ will be represented as $\omega(f)$. From now on, we restrict ourselves to the case of $\{0, 1, 2\}$-dominating functions. We will identify f with the three subsets of $V(G)$ induced by f: $V_i = \{v \in V(G) : f(v) = i\}$ for every $i \in \{0, 1, 2\}$ and also, we define a $\{0, 1, 2\}$-dominating function f as $f(V_0, V_1, V_2)$. In that sense, the weight of f is $\omega(f) = 2|V_2| + |V_1|$.

In recent years, among the most studied $\{0, 1, 2\}$-dominating functions are the "Italian dominating functions", which were introduced in [4] under the name of "Roman $\{2\}$-dominating functions". An *Italian dominating function* (IDF) is a $\{0, 1, 2\}$-dominating function $f(V_0, V_1, V_2)$ satisfying the condition that for every $v \in V_0$, $f(N(v)) \geq 2$, i.e., either $|N(v) \cap V_1| \geq 2$ or $|N(v) \cap V_2| \geq 1$. Clearly, $V_1 \cup V_2$ is a dominating set, and if $V_2 = \emptyset$, then V_1 is a 2-dominating set of G. The *Italian domination number* of G, denoted by $\gamma_I(G)$, is the minimum weight among all IDFs on G. An IDF of weight $\gamma_I(G)$ is called a $\gamma_I(G)$-function. A similar agreement will be assumed when referring to optimal functions (and sets) associated to other parameters used in the paper. This parameter and some of their variants was further studied in [2, 3, 8, 9, 13, 14, 15, 18].

In this article we introduce a new variant of this concept, which is a combination between Italian domination and maximal domination. A *starred Italian dominating function* (SIDF) is an IDF $f(V_0, V_1, V_2)$ such that $V_1 \cup V_2$ is a maximal dominating set of G, i.e., V_0 is not a dominating set of G. Observe that for some vertex $v \in V_1$, $N(v) \subseteq V_1 \cup V_2$ and the subgraph induced by $N[v]$ has a spanning star centred at v. The *starred Italian domination number* of G, denoted by $\gamma_I^*(G)$, is the minimum weight among all SIDFs on G.

The article is structured as follows. In Section 2 we define additional notation and expose some preliminary results. Section 3 is devoted to the study of relationships between the starred Italian domination number and other domination parameters. Finally, in Section 4 we compute exact formulas for some families of graphs and show that the problem of finding the starred Italian domination number of a graph is NP-hard.

2. Notation and preliminary results

Throughout the article, we will use the following notation. Given a graph G and a set of vertices D, the *open neighbourhood* and *closed neighbourhood* of D is $N(D) = \bigcup_{v\in D} N(v)$ and $N[D] = N(D) \cup D$, respectively. We denote the *degree* of vertex v by $\delta(v) = |N(v)|$. We say that a vertex $v \in V(G)$ is *universal* if $\delta(v) = |V(G)| - 1$.

Let $D \subseteq V(G)$ and $v \in D$. The *private neighbourhood* $pn(v, D)$ of v is defined by $pn(v, D) = \{u \in V(G) : N(u) \cap D = \{v\}\}$. Each vertex in $pn(v, D)$ is called a *private neighbour* of v. The *external private neighbourhood* $epn(v, D)$ consists of those private neighbours of v belonging to $V(G) \setminus D$. Thus, $epn(v, D) = pn(v, D) \setminus D$. The subgraph of G induced by $D \subseteq V(G)$ is denoted by $G[D]$.

A *leaf* of G is a vertex of degree one. A *support* vertex of G is the vertex adjacent to a leaf, a *weak support* vertex is a support vertex that is adjacent to exactly one leaf. The set of leaves and support vertices are denoted by $L(G)$ and $S(G)$, respectively. We will use the notation N_n, K_n, $K_{1,n-1}$, C_n, P_n, $K_{r,n-r}$ and $W_n = N_1 + C_{n-1}$ for empty graphs, complete graphs, star graphs, cycle graphs, path graphs, complete bipartite graphs and the wheel graphs of order n, respectively. A subdivided star graph, denoted by $K^*_{1,(n-1)/2}$, is a graph of order n (odd) obtained from a star $K_{1,(n-1)/2}$ by subdividing every edge exactly once.

We assume that the reader is familiar with the basic concepts, notation, and terminology in graphs. If this is not the case, we suggest the books [11, 12].

Let G be a disconnected graph and let $G_1,\ldots,G_r$ with $r \geq 2$, be the components of G. Observe that any $\gamma^*_I(G)$-function is formed by an SIDF on one component G_j and IDFs on the rest of the components different from G_j. Thus, the following result for the case of disconnected graphs is obtained.

Observation 2.1. *Let $G_1,\ldots,G_r$ with $r \geq 2$, be the components of a disconnected graph G. Then*

$$\gamma^*_I(G) = \min_{1\leq i,j\leq r} \left\{ \gamma^*_I(G_j) + \sum_{i=1, i\neq j}^{r} \gamma_I(G_i) \right\}.$$

As a consequence of the observation above, throughout this article we only consider the study of SIDFs on connected graphs.

We next establish some properties satisfied by $\gamma^*_I(G)$-functions.

Observation 2.2. *Let G be a connected graph. If $f(V_0, V_1, V_2)$ is a $\gamma^*_I(G)$-function, then the following statements hold.*

(i) $V(G) \setminus N[V_0]$ *is a nonempty subset of* V_1.
(ii) $epn(v, V_1 \cup V_2) \neq \emptyset$, *for every* $v \in V_2$.

Proof. Let $v \in V(G) \setminus N[V_0]$. If $v \in V_2$, then the function $f'(V_0', V_1', V_2')$ defined by $V_2' = V_2 \setminus \{v\}$, $V_1' = V_1 \cup \{v\}$ and $V_0' = V_0$, is an SIDF on G of weight $\omega(f') = \gamma_I^*(G) - 1$, which is a contradiction. Therefore, $v \in V_1$ and (i) follows.

In order to prove (ii), we suppose that there exists $v \in V_2$ such that $epn(v, V_1 \cup V_2) = \emptyset$. Notice that the function f'', defined by $f''(v) = 1$ and $f''(x) = f(x)$ otherwise, is an SIDF on G of weight $\omega(f'') = \gamma_I^*(G) - 1$, which is a contradiction. Therefore, $epn(v, V_1 \cup V_2) \neq \emptyset$ for every $v \in V_2$, as desired. □

Proposition 2.3. *For any connected graph* G*, there exists a* $\gamma_I^*(G)$*-function* $f(V_0, V_1, V_2)$ *such that either* $V_2 = \emptyset$ *or every vertex* $v \in V_2$ *satisfies that* $|epn(v, V_1 \cup V_2)| \geq 2$.

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function such that $|V_2|$ is minimum among all $\gamma_I^*(G)$-functions. If $V_2 = \emptyset$, then we are done. Hence, let $v \in V_2$. By Observation 2.2 we have that $|epn(v, V_1 \cup V_2)| \geq 1$. Suppose that $epn(v, V_1 \cup V_2) = \{u\}$. Notice that the function $f'(V_0', V_1', V_2')$, defined by $V_0' = V_0 \setminus \{u\}$, $V_1' = V_1 \cup \{v, u\}$ and $V_2' = V_2 \setminus \{v\}$), is a $\gamma_I^*(G)$-function and $|V_2'| < |V_2|$, which is a contradiction. Therefore, $|epn(v, V_1 \cup V_2)| \geq 2$, and the proof is complete. □

3. General results

We begin this section with a result that relates the Italian domination number and the starred Italian domination number of a graph.

Theorem 3.1. *For any connected graph* G,

$$\gamma_I(G) \leq \gamma_I^*(G) \leq \gamma_I(G) + \delta(G).$$

In addition, $\gamma_I^*(G) = \gamma_I(G)$ *if and only if there exists a* $\gamma_I(G)$*-function assigning the value* 1 *to a weak support vertex and its leaf.*

Proof. The lower bound follows directly from the fact that every SIDF is an IDF. In order to prove the upper bound, we consider a vertex v of minimum degree and let $f(V_0, V_1, V_2)$ be a $\gamma_I(G)$-function. Notice that the function $f'(V_0', V_1', V_2')$, defined by $V_1' = V_1 \cup (N[v] \cap V_0)$ and $V_2' = V_2$, is an SIDF on G. Also, as $N[v] \cap (V_1 \cup V_2) \neq \emptyset$, it follows that $|N[v] \cap V_0| \leq \delta(v)$. Therefore, $\gamma_I^*(G) \leq \omega(f') = |V_1'| + 2|V_2'| \leq |V_1| + 2|V_2| + \delta(v) = \gamma_I(G) + \delta(G)$, as desired.

Now we consider equality. First, we suppose that $\gamma_I^*(G) = \gamma_I(G)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function. So, f is also a $\gamma_I(G)$-function. Since V_0 is not a dominating set of G, by Observation 2.2 there exists a vertex $h \in V_1 \setminus N(V_0)$. If $\delta(h) \geq 2$ or $N(h) \cap V_2 \neq \emptyset$, then as $N(h) \subseteq V_1 \cup V_2$, the function $f'(V_0', V_1', V_2')$, defined by $V_0' = V_0 \cup \{h\}$, $V_1' = V_1 \setminus \{h\}$ and $V_2' = V_2$, is an IDF on G of weight $\omega(f') = \omega(f) - 1 = \gamma_I^*(G) - 1 = \gamma_I(G) - 1$, which is a contradiction. Hence, h is a leaf (recall that $f(h) = 1$ by assumption) adjacent to a weak support vertex, namely u, which satisfies that $f(u) = 1$, as desired.

On the other hand, if there exists a $\gamma_I(G)$-function g satisfying $g(u) = g(h) = 1$, where u is a weak support vertex and h its leaf vertex, then g is an SIDF on G. Therefore, $\gamma_I(G) \le \gamma_I^*(G) \le \omega(g) = \gamma_I(G)$. Therefore, $\gamma_I^*(G) = \gamma_I(G)$, which completes the proof. □

The upper bound above is tight. For instance, it is achieved for any star graph $K_{1,n-1}$, with $n \ge 3$.

Corollary 3.2. *Let G be a connected graph. If $\delta(G) \ge 2$, then $\gamma_I^*(G) \ge \gamma_I(G) + 1$.*

Now, we relate the starred Italian domination number with the classical domination number. For this purpose, we shall need the following result.

Theorem 3.3 ([10])**.** *If G is a nontrivial connected graph with $\gamma_2(G) = \gamma(G)$, then $\delta(G) \ge 2$.*

Theorem 3.4. *For any nontrivial connected graph G,*

$$\gamma_I^*(G) \ge \gamma(G) + 1.$$

Furthermore, if $\gamma_I^*(G) = \gamma(G) + 1$, then

$$\gamma_2(G) = \begin{cases} \gamma(G) + 1 & \text{if } \delta(G) = 1, \\ \gamma(G) & \text{if } \delta(G) \ge 2. \end{cases}$$

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function. By Observation 2.2 (i) we deduce that $V_1 \setminus N(V_0) \neq \emptyset$. Notice that for every vertex $v \in V_1 \setminus N(V_0)$, the set $(V_1 \cup V_2) \setminus \{v\}$ is a dominating set of G. Hence,

$$\gamma_I^*(G) = 2|V_2| + |V_1| \ge |V_2| + |V_1| = |(V_1 \cup V_2) \setminus \{v\}| + 1 \ge \gamma(G) + 1,$$

and the lower bound follows.

Now, we assume that $\gamma_I^*(G) = \gamma(G) + 1$. Hence, we have equalities in inequality chain above, in particular, we obtain that $|V_2| = 0$. Thus, V_1 is a 2-dominating set of G, which implies that $\gamma_2(G) \le |V_1| = \gamma_I^*(G) = \gamma(G)+1$. If $\delta(G) = 1$, then Theorem 3.3 leads to $\gamma_2(G) = \gamma(G) + 1$.

On the other hand, we consider that $\delta(G) \ge 2$ and suppose that $\gamma_2(G) = \gamma(G) + 1$. This implies that V_1 is a $\gamma_2(G)$-set. Since $\delta(G) \ge 2$ then $V_1 \setminus \{v\}$ is a 2-dominating set of cardinality $\gamma_2(G) - 1$, which is a contradiction. Therefore, $\gamma_2(G) = \gamma(G)$, and the result follows. □

There exist graphs G with $\delta(G) = 1$ and $\gamma_2(G) = \gamma(G) + 1$ such that $\gamma_I^*(G) > \gamma(G) + 1$. For example, the subdivided star $K^*_{1,(n-1)/2}$ satisfies $\frac{n+3}{2} = \gamma_I^*(K^*_{1,(n-1)/2}) > \gamma_2(K^*_{1,(n-1)/2}) = \frac{n+1}{2} = \gamma(K^*_{1,(n-1)/2}) + 1$. Analogously, we next show that there exist graphs G with $\delta(G) \ge 2$ and $\gamma_2(G) = \gamma(G)$ such that $\gamma_I^*(G) > \gamma(G)+1$. The following graph G_k satisfies that $\gamma_2(G_k) = \gamma(G_k) = 4$ and $\delta(G_k) \ge 2$, but $\gamma_I^*(G_k) = 6 > \gamma(G_k) + 1$. The graph G_k is constructed from two complete bipartite graphs $K_{2,k}$ (with

$k \geq 4$) by adding k new edges which form a matching between the vertices of degree 2 of each $K_{2,k}$. Figure 1 shows the graph G_4.

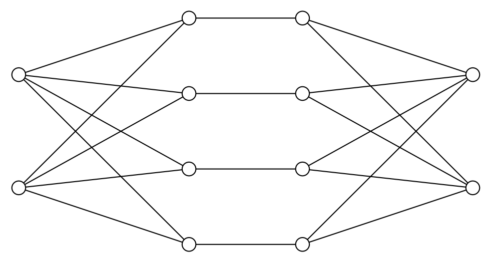

FIGURE 1. The graph G_4.

Next, we give a sufficient condition for a graph G with $\gamma_2(G) = \gamma(G)$ to satisfy $\gamma_I^*(G) = \gamma(G) + 1$.

Theorem 3.5. *Let G be a nontrivial connected graph such that $\gamma_2(G) = \gamma(G)$. If there exists a $\gamma_2(G)$-set D such that $G[V(G) \setminus D]$ has an isolated vertex, then $\gamma_I^*(G) = \gamma(G) + 1$.*

Proof. Let D be a $\gamma_2(G)$-set such that v is an isolated vertex of $G[V(G) \setminus D]$. Let $f(V_0, V_1, V_2)$ be a function defined by $V_0 = V(G) \setminus (D \cup \{v\})$ and $V_1 = D \cup \{v\}$. Since $N(v) \subseteq D$ and every vertex that is not in D has at least two neighbours in D, it follows that f is an SIDF on G. Therefore, $\gamma_I^*(G) \leq \omega(f) = |V_1| = |D \cup \{v\}| = \gamma_2(G) + 1 = \gamma(G) + 1$, and by Theorem 3.4 the result follows. □

Next, we give a relationship between the starred Italian domination number, the maximal domination number and the domination number.

Theorem 3.6. *For any connected graph G,*

$$\gamma_m(G) \leq \gamma_I^*(G) \leq \gamma_m(G) + \gamma(G).$$

Furthermore, if $\gamma_I^*(G) = \gamma_m(G)$, then every $\gamma_I^*(G)$-function $f(V_0, V_1, V_2)$ satisfies $V_2 = \emptyset$.

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function. Since $V_1 \cup V_2$ is a maximal dominating set of G, we have that $\gamma_m(G) \leq |V_1 \cup V_2| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_I^*(G)$, and the lower bound follows. If $\gamma_I^*(G) = \gamma_m(G)$, then we have equalities in the inequality chain above, and as a consequence, $V_2 = \emptyset$.

In order to prove the upper bound, let S be a $\gamma_m(G)$-set and D a $\gamma(G)$-set. Since $V(G) \setminus S$ is not a dominating set of G, there exists a vertex $v \in S$ such that $N(v) \subseteq S$. Notice that the function $f'(V_0', V_1', V_2')$, defined by $V_2' = (D \cap S) \setminus \{v\}$ and $V_1' = (D \cup S) \setminus V_2'$, is an SIDF on G. Therefore, $\gamma_I^*(G) \leq \omega(f') = 2|V_2'| + |V_1'| \leq |S| + |D| = \gamma_m(G) + \gamma(G)$, as desired. □

The upper bound given in Theorem 3.6 is tight. For instance, it is achieved for any star graph $K_{1,n-1}$, with $n \geq 3$.

A *vertex cover* of a graph G is a set of vertices such that each edge is incident to at least one vertex of this set. The minimum cardinality among all vertex covers is denoted by $\beta(G)$. A vertex cover of cardinality $\beta(G)$ is called $\beta(G)$-*set*.

Theorem 3.7. *For any connected graph G,*

$$\gamma_I^*(G) \leq \beta(G) + |S(G)| + 1.$$

Proof. Let D be a $\beta(G)$-set containing all support vertices of G and fix a vertex $v \in V(G) \setminus D$. We claim that the function $f(V_0, V_1, V_2)$, defined by $V_1 = (D \setminus S(G)) \cup \{v\}$ and $V_2 = D \cap S(G)$, is an SIDF on G. Let $x \in V_0$. If $x \in L(G)$, then $f(N(x)) = 2$ since $S(G) \subseteq V_2$. Moreover, if $x \notin L(G)$, then as $V(G) \setminus D$ is an independent set, it follows that $N(x) \subseteq D$, and as a consequence, $f(N(x)) \geq 2$. Hence, f is an SIDF on G, as desired. Therefore, $\gamma_I^*(G) \leq \omega(f) = |V_1| + 2|V_2| = |(D \setminus S(G)) \cup \{v\}| + 2|D \cap S(G)| = \beta(G) + |S(G)| + 1$, which completes the proof. □

The following result is an immediate consequence of Theorems 3.4 and 3.7.

Theorem 3.8. *For any connected graph G with $\delta(G) \geq 2$,*

$$\gamma_I^*(G) \leq \beta(G) + 1.$$

Furthermore, if $\gamma(G) = \beta(G)$, then $\gamma_I^*(G) = \beta(G) + 1$.

There exist graphs G with $\delta(G) \geq 2$ and $\gamma_I^*(G) = \beta(G) + 1$ such that $\gamma(G) < \beta(G)$. For instance, we consider the graph $K_2 + N_r$, where $r \geq 1$. In this case, we have that $\gamma_I^*(K_2 + N_r) = 3 = \beta(K_2 + N_r) + 1$ and $\gamma(K_2 + N_r) = 1 < 2 = \beta(K_2 + N_r)$.

The following results provide bounds for the starred Italian domination number in terms of the order, the minimum degree and the maximum degree of G.

Theorem 3.9. *For any connected graph G,*

$$\gamma_I^*(G) \geq \delta(G) + 1.$$

Furthermore, the following statements are equivalent.

(i) $\gamma_I^*(G) = \delta(G) + 1$.
(ii) *There exists a vertex $v \in V(G)$ of minimum degree such that the function $f(V(G) \setminus N[v], N[v], \emptyset)$ is a $\gamma_I^*(G)$-function.*

Proof. By Observation 2.2 (i) the bound follows. Suppose that (i) holds and let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function. By Observation 2.2 (i) there exists a vertex $v \in V_1 \setminus N(V_0)$, i.e, $N[v] \subseteq V_1 \cup V_2$. This implies that $\delta(G) + 1 \leq |N[v]| \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_I^*(G) = \delta(G) + 1$. Hence, we have equalities in the inequality chain above. In particular, $V_2 = \emptyset$ and so,

$|V_1| = |N[v]| = \delta(G) + 1$. Therefore v is a vertex of minimum degree and $f(V(G) \setminus N[v], N[v], \emptyset)$ is a $\gamma_I^*(G)$-function. Thus, Statement (ii) follows. The other implication is straightforward, which completes the proof. $\square$

Theorem 3.10. *The following statements hold for any connected graph G of order n.*

(i) *If $diam(G) = 2$, then $\gamma_I^*(G) \leq 2\delta(G) + 1$.*
(ii) *If $diam(G) \geq 3$, then $\gamma_I^*(G) \leq n - \delta(G)$.*

Proof. If $diam(G) = 2$ and v is a vertex of minimum degree of G, then $N(v)$ is a dominating set and $N[v]$ is a maximal dominating set. Hence, by Theorem 3.6 we have that $\gamma_I^*(G) \leq \gamma_m(G) + \gamma(G) \leq |N[v]| + |N(v)| = 2\delta(G) + 1$, which completes the proof of (i).

We now assume that $diam(G) \geq 3$ and let $v_1 v_2 \cdots v_{diam(G)+1}$ be a diametrical path of G. Since $N(v_1) \cap L(G) = \emptyset$ we deduce that the function $f(V_0, V_1, V_2)$, defined by $V_0 = N(v_1)$ and $V_1 = V(G) \setminus N(v_1)$, is an SIDF on G. Hence $\gamma_I^*(G) \leq \omega(f) = n - |N(v_1)| \leq n - \delta(G)$, which completes the proof. $\square$

The bounds above are tight. The bound given in (i) is achieved for any star graph $K_{1,n-1}$, with $n \geq 3$, while the bound given in (ii) is achieved for the path P_4 and the cycle C_6.

Theorem 3.11. *If G is a connected graph of order n, then*

$$\gamma_I^*(G) \geq \left\lceil \frac{2n + \delta(G) + \Delta(G)}{\Delta(G) + 2} \right\rceil.$$

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function and let $V_{0,2} = \{x \in V_0 : N(x) \cap V_2 \neq \emptyset\}$ and $V_{0,1} = V_0 \setminus V_{0,2}$. By Observation 2.2 (i) there exists a vertex $u \in V_1 \setminus N(V_0)$.

In order to prove the bound, we consider the following sets: $V_2^u = V_2 \cap N(u)$ and $V_1^u = V_1 \cap N(u)$. Observe that $|V_1^u| + |V_2^u| = \delta(u) \geq \delta(G)$. Since every vertex in $V_2 \setminus V_2^u$ can have at most $\Delta(G)$ neighbours in $V_{0,2}$, we obtain that $|V_{0,2}| \leq \Delta(G)|V_2| - |V_2^u|$. Also, since every vertex in $V_{0,1}$ has at least two neighbours in $V_1 \setminus \{u\}$ and every vertex in $V_1 \setminus (V_1^u \cup \{u\})$ has at most $\Delta(G)$ neighbours in $V_{0,1}$, we deduce that $2|V_{0,1}| \leq \Delta(G)(|V_1| - 1) - |V_1^u|$. Hence,

$$\begin{aligned}
n &= |V_{0,1}| + |V_{0,2}| + |V_1| + |V_2| \\
&\leq (\Delta(G)(|V_1| - 1) - |V_1^u|)/2 + (\Delta(G)|V_2| - |V_2^u|) + |V_1| + |V_2| \\
&= (\Delta(G) + 2)|V_1|/2 + (\Delta(G) + 1)|V_2| - (\Delta(G)/2 + |V_1^u|/2 + |V_2^u|) \\
&\leq (\Delta(G) + 2)(|V_1|/2 + |V_2|) - (\Delta(G) + |V_1^u| + |V_2^u|)/2 \\
&\leq (\Delta(G) + 2)\gamma_I^*(G)/2 - (\Delta(G) + \delta(G))/2.
\end{aligned}$$

Therefore, $\gamma_I^*(G) \geq \left\lceil \frac{2n+\delta(G)+\Delta(G)}{\Delta(G)+2} \right\rceil$. $\square$

We now proceed to construct a family of graphs for which the lower bound above is attained. For any integer $k \geq 5$, let H_k be the graph with vertex set $V(H_k) = \{v, u_1, u_2, u_3, w_1, \ldots, w_{k-4}\}$ and edge set $E(H_k) = (\bigcup_{i=1}^{k-4}\{u_1w_i, u_2w_i, u_3w_i\}) \cup \{vu_1, vu_3, u_2u_1, u_2u_3\}$. Notice that $|V(H_k)| = k = \Delta(H_k) + 2$, $\delta(H_k) = 2$ and the function $f(V_0, V_1, V_2)$, defined by $V_2 = \emptyset$, $V_1 = \{v, u_1, u_3\}$ and $V_0 = V(H_k) \setminus V_1$, is a $\gamma_I^*(H_k)$-function, which implies that $\gamma_I^*(H_k) = 3$. Therefore, the lower bound is achieved for the graph H_k. Figure 2 shows the graph H_7.

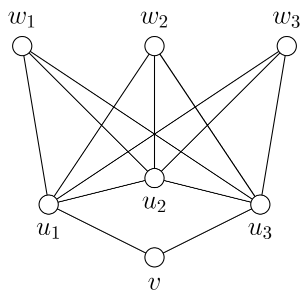

FIGURE 2. The graph H_7.

We now proceed to characterize all graphs G of order $n \geq 3$ achieving the limit cases of the trivial bounds $3 \leq \gamma_I^*(G) \leq n$.

Proposition 3.12. *If G is a connected graph of order $n \geq 3$, then the following statements hold.*

(i) $\gamma_I^*(G) = 3$ *if and only if one of the following holds.*
  a) $\gamma(G) = \delta(G) = 1$.
  b) *There exist a $\gamma_2(G)$-set S and a vertex $v \in V(G) \setminus S$ such that $|S| = 2$ and $N(v) \subseteq S$.*

(ii) $\gamma_I^*(G) = n$ *if and only if G is the complete graph K_n or the path P_3.*

Proof. We first proceed to prove (i). Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function and suppose that $\omega(f) = 2|V_2| + |V_1| = 3$. Recall that $V_1 \cup V_2$ is a maximal dominating set, which implies that there exists a vertex $v \in V_1$ such that $N(v) \subseteq V_1 \cup V_2$ and $(V_1 \cup V_2) \setminus \{v\}$ is a dominating set. We analyse the next two cases.

CASE 1: $|V_2| = 1$.

In this case, $V_1 = \{v\}$ and V_2 is a dominating set, which implies that v is a leaf vertex. Hence $\gamma(G) = \delta(G) = 1$, and a) follows.

CASE 2: $|V_2| = 0$.

In this case, $|V_1| = \{u, v, w\}$ and $N(v) \subseteq \{u, w\}$. By definition we deduce that $\{u, w\}$ is a 2-dominating set and as $\gamma_2(G) \geq 2$, we obtain that $\{u, w\}$ is a $\gamma_2(G)$-set that satisfies conditions of b).

On the other hand, we consider that G satisfies one of the two conditions above. To conclude that $\gamma_I^*(G) = 3$, we proceed to show how to construct an SIDF $f(V_0, V_1, V_2)$ on G of weight three for each of the two conditions.

a) Let v be a universal vertex and u be a leaf of G. In this case $V_1 = \{u\}$ and $V_2 = \{v\}$.
b) In this case, $V_1 = S \cup \{v\}$ and $V_2 = \emptyset$.

It is straightforward that in all cases f is an SIDF of G. Therefore $\gamma_I^*(G) = 3$, which completes the proof of (i).

Finally, we proceed to prove (ii). If G is isomorphic to P_3 or K_n, then is straightforward that $\gamma_I^*(G) = |V(G)| = n$. Conversely, assume that G is a connected graph of order n such that $\gamma_I^*(G) = n$. By Theorem 3.10 we deduce that $diam(G) \leq 2$. If $diam(G) = 1$, then G is isomorphic to K_n. From now on we assume that $diam(G) = 2$. Let $v_1v_2v_3$ be a diametrical path of G. If $|N(v_1)| \geq 2$, then the function $f(V_0, V_1, V_2)$, defined by $V_0 = \{v_1\}$ and $V_1 = V(G) \setminus \{v_1\}$, is an SIDF on G and $\omega(f) = n - 1$, which is a contradiction. Hence, $\delta(v_1) = 1$ and $\delta(v_2) = n - 1$. So, by statement (i) item a) we deduce that $\gamma_I^*(G) = 3 = n$. Therefore, G is isomorphic to P_3 and the result follows. □

4. Exact formulas for some families of graphs and computational complexity

We begin this section by giving the starred Italian domination number of paths and cycles. For this purpose, we shall need the following tools.

Observation 4.1. *Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G)$-function. If $u, v \in V_1 \cup V_2$ are two non-adjacent vertices, then*

$$\gamma_I^*(G + uv) \leq \gamma_I^*(G).$$

If $f(V_0, V_1, V_2)$ is a $\gamma_I^*(P_n)$-function, then $N(v) \subseteq V_0$ for every $v \in V_2$. In addition, if we assume that $|V_2|$ is minimum, then it is easy to deduce that $V_2 = \emptyset$. This implies that $L(P_n) \subseteq V_1$. Hence, Observation 4.1 leads to the following result.

Lemma 4.2. *For any integer $n \geq 3$,*

$$\gamma_I^*(C_n) \leq \gamma_I^*(P_n).$$

The following result provides the starred Italian domination number of paths and cycles.

Theorem 4.3. *For any integer $n \geq 3$,*

$$\gamma_I^*(C_n) = \gamma_I^*(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Proof. Let $P_n = v_1 \ldots v_n$ be the path of order $n \geq 3$, and we consider the set

$$D = \bigcup_{i=1}^{\lfloor (n+1)/2 \rfloor} \{v_{2i-1}\} \cup \{v_{n-1}, v_n\}.$$

Notice that the function $f(V_0, V_1, V_2)$, defined by $V_1 = D$ and $V_0 = V(P_n) \setminus D$, is an SIDF on P_n. Hence, $\gamma_I^*(P_n) \leq \omega(f) = |D| = \lfloor \frac{n+1}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil + 1$.

Therefore, by Theorem 3.11 and Lemma 4.2 we have that $\lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n}{2} + 1 \rceil \leq \gamma_I^*(C_n) \leq \gamma_I^*(P_n) \leq \lceil \frac{n}{2} \rceil + 1$. Thus, we have equalities in the inequality chain above, which implies that the result follows. □

We next compute the starred Italian domination number for the join graph of two nontrivial graphs.

Theorem 4.4. *For any nontrivial graphs G_1 and G_2 of order n_1 and n_2 respectively,*

$$\gamma_I^*(G_1 + G_2) = 1 + \min\{n_1 + \delta(G_2), n_2 + \delta(G_1)\}.$$

Proof. Let v be a vertex of minimum degree in $G_1 + G_2$. Since $n_i \geq 2$ for $i \in \{1, 2\}$, it follows that the function $f(V_0, V_1, V_2)$, defined by $V_1 = N[v]$ and $V_0 = V(G_1 + G_2) \setminus N[v]$, is an SIDF on $G_1 + G_2$. Hence, $\gamma_I^*(G_1 + G_2) \leq \omega(f) = |N[v]| = \delta(G_1 + G_2) + 1$. By Theorem 3.9 and the well-known equality $\delta(G_1 + G_2) = \min\{n_1 + \delta(G_2), n_2 + \delta(G_1)\}$, the result follows. □

Corollary 4.5. *For any integers n, r such that $n - r \geq r \geq 2$,*

$$\gamma_I^*(K_{r,n-r}) = r + 1.$$

Now, we give the exact value for $\gamma_I^*(N_1 + G)$. For this purpose, we need to introduce the following definition on a graph G.

$$\mathcal{N}_{\delta,\gamma}(G) = \{v \in V(G) : |N(v)| = \delta(G) \text{ and } N[N(v)] = V(G)\}.$$

So, $\mathcal{N}_{\delta,\gamma}(G)$ is the set of all vertices of minimum degree of G with eccentricity 1 if G is a complete graph or eccentricity 2 otherwise.

Theorem 4.6. *For any graph G,*

$$\gamma_I^*(N_1 + G) = \begin{cases} \delta(G) + 2 & \text{if } \mathcal{N}_{\delta,\gamma}(G) \neq \emptyset, \\ \delta(G) + 3 & \text{otherwise.} \end{cases}$$

Proof. Let $V(N_1) = \{u\}$. As $\delta(N_1 + G) = \delta(G) + 1$, let $v \in V(G)$ be a vertex of minimum degree in $N_1 + G$. Notice that the function $f(V_0, V_1, V_2)$, defined by $V_2 = \{u\}$, $V_1 = N[v] \setminus \{u\}$ and $V_0 = V(N_1 + G) \setminus N[v]$, is an SIDF on $N_1 + G$. By Theorem 3.9 and statement above we have that $\delta(G) + 2 = \delta(N_1 + G) + 1 \leq \gamma_I^*(N_1 + G) \leq \omega(f) = \delta(G) + 3$. Theorem 3.9 leads to $\gamma_I^*(N_1 + G) = \delta(G) + 2$ if and only if $\mathcal{N}_{\delta,\gamma}(G) \neq \emptyset$, which completes the proof. □

Corollary 4.7. *The following equalities hold for any integer $n \geq 4$.*

(i) $\gamma_I^*(K_{1,n-1}) = 3$.

(ii) $\gamma_I^*(W_n) = \begin{cases} 4 & \text{if } n \in \{4, 5, 6\}, \\ 5 & \text{otherwise.} \end{cases}$

Let G_1 and G_2 be two graphs. The corona product graph $G_1 \odot G_2$ is defined as the graph obtained from G_1 and G_2, by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining by an edge every vertex from the i^{th} - copy of G_2 with the i^{th}-vertex of G_1. For every $x \in V(G_1)$, G_2^x will denote the copy of G_2 in $G_1 \odot G_2$ associated with x.

Next, we study the starred Italian domination number of any corona product graph. First, we obtain $\gamma_I^*(G_1 \odot G_2)$, for any nontrivial graph G_2.

Theorem 4.8. *Let G_1 be a connected graph of order n_1. If G_2 is a nontrivial graph, then*

$$\gamma_I^*(G_1 \odot G_2) = \begin{cases} 2n_1 + \delta(G_2) & \text{if } \mathcal{N}_{\delta,\gamma}(G_2) \neq \emptyset, \\ 2n_1 + \delta(G_2) + 1 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 4.6 we only need to prove the equality $\gamma_I^*(G_1 \odot G_2) = 2(n_1 - 1) + \gamma_I^*(N_1 + G_2)$.

We first proceed to prove that $\gamma_I^*(G_1 \odot G_2) \leq 2(n_1 - 1) + \gamma_I^*(N_1 + G_2)$. From some vertex $y \in V(G_1)$ and any $\gamma_I^*(N_1 + G_2)$-function f, we define a function g on $G_1 \odot G_2$ as follows. For every vertex $x \in V(G_1) \setminus \{y\}$, we set $g(x) = 2$ and $g(V(G_2^x)) = 0$, and the restriction of g to $V(G_2^y) \cup \{y\}$ is induced by f. Notice that g is an SIDF on $G_1 \odot G_2$, which implies that $\gamma_I^*(G_1 \odot G_2) \leq \omega(g) = 2|V(G_1) \setminus \{y\})| + \omega(f) = 2(n_1 - 1) + \gamma_I^*(N_1 + G_2)$.

Finally, we prove that $\gamma_I^*(G_1 \odot G_2) \geq 2(n_1 - 1) + \gamma_I^*(N_1 + G_2)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_I^*(G_1 \odot G_2)$-function. Since f is an IDF we deduce that $f(V(G_2^x) \cup \{x\}) \geq 2$ for every $x \in V(G_1)$. Moreover, by Observation 2.2 there exists a vertex $y \in V(G_1)$ such that f restricted to $V(G_2^y) \cup \{y\}$ is an SIDF on $G_1 \odot G_2[V(G_2^y) \cup \{y\}]$. Hence, we obtain that $f(V(G_2^y) \cup \{y\}) \geq \gamma_I^*(G_1 \odot G_2[V(G_2^y) \cup \{y\}]) = \gamma_I^*(N_1 + G_2)$, and as a consequence,

$$\begin{aligned} \gamma_I^*(G_1 \odot G_2) &= \sum_{x \in V(G_1)} (f(x) + f(V(G_2^x))) \\ &= \sum_{x \in V(G_1) \setminus \{y\}} (f(x) + f(V(G_2^x))) + (f(y) + f(V(G_2^y))) \\ &\geq 2(n_1 - 1) + \gamma_I^*(N_1 + G_2). \end{aligned}$$

Therefore, the result follows. □

Now, we compute the starred Italian domination number of $G \odot N_1$. For this purpose, we shall need the following result given in [15].

Theorem 4.9 ([15])**.** *For any graph G of order n and any graph G',*

$$\gamma_I(G \odot G') = \begin{cases} n + \gamma(G) & \text{if } G' \cong N_1, \\ 2n & \text{otherwise.} \end{cases}$$

Theorem 4.10. *For any connected graph G of order n,*

$$\gamma_I^*(G \odot N_1) = n + \gamma(G).$$

Proof. For any $\gamma(G)$-set D, we consider the function $f(V_0, V_1, V_2)$ defined by $V_1 = D \cup L(G \odot N_1)$, $V_2 = \emptyset$ and $V_0 = V(G) \setminus D$. Notice that f is an SIDF on $G \odot N_1$. Hence, $\gamma_I^*(G \odot N_1) \leq |V_1| = n + \gamma(G)$ (see Figure 3 for an example).

Finally, by Theorems 3.1 and 4.9 we deduce that $\gamma_I^*(G \odot N_1) \geq \gamma_I(G \odot N_1) = n + \gamma(G)$. Therefore, $\gamma_I^*(G \odot N_1) = n + \gamma(G)$, which completes the proof. □

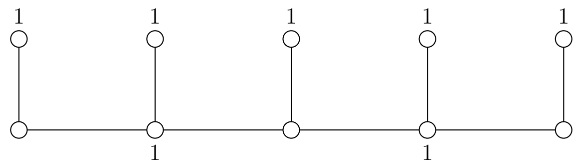

FIGURE 3. Graph $P_5 \odot N_1$, with the labelling used in the proof of Theorem 4.10 (vertices with no drawn label have label zero).

The Domination Problem is an NP-complete decision problem [6], even when restricted to bipartite graphs [5] or chordal graphs [1]. Hence, the optimization problem of computing the domination number of a graph is NP-hard. Using this fact and the theorem above, we next study the complexity issue for the starred Italian domination number of a graph.

Theorem 4.11. *The problem of computing the starred Italian domination number of a graph is NP-hard.*

Proof. Let G be a connected graph. Theorem 4.10 states that $\gamma_I^*(G \odot N_1) = |V(G)| + \gamma(G)$. Hence, the problem of computing the starred Italian domination number has the same computational complexity as the domination number of G, which is known to be NP-Hard. $\square$

5. Open problems

(1) According to Theorem 3.1, for any tree T it follows that $\gamma_I(T) \leq \gamma_I^*(T) \leq \gamma_I(T) + 1$. Can we characterize the families of trees for which $\gamma_I^*(T) = \gamma_I(T)$ or $\gamma_I^*(T) = \gamma_I(T) + 1$?
(2) We have shown that $\gamma_I^*(G) \geq \gamma(G) + 1$ and we have also given some necessary and sufficient conditions for the graphs which satisfy the equality. We propose the problem of characterizing all graphs for which this equality holds.
(3) We propose to study the starred Italian domination number of other product graphs.
(4) Since the problem of finding $\gamma_I^*(G)$ is NP-hard, we consider the next question. Is there a polynomial-time algorithm for finding $\gamma_I^*(G)$ for some specific families of graphs?

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