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ORIENTED UNICYCLIC GRAPHS WITH MINIMAL SKEW RANDIĆ ENERGY

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ABSTRACT. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and G^σ be an orientation of G. Denote by $d(v_i)$ the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The skew Randić matrix of G^σ, denoted by $R_S(G^\sigma)$, is the real skew-symmetric matrix $(r_{ij})_{n\times n}$, where $r_{ij} = 1/\sqrt{d(v_i)d(v_j)}$ and $r_{ji} = -1/\sqrt{d(v_i)d(v_j)}$ if $v_i \to v_j$ is an arc of G^σ, otherwise $r_{ij} = r_{ji} = 0$. The skew Randić energy $\mathcal{RE}_S(G^\sigma)$ of G^σ is defined as the sum of the norms of all the eigenvalues of $R_S(G^\sigma)$. In this paper, the oriented unicyclic graphs with minimal skew Randić energy are determined.

1. INTRODUCTION

All graphs considered in this paper are finite and simple (i.e., without loops and multiple edges). Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $d_G(v_i)$ (or $d(v_i)$ for short) be the degree of the vertex v_i of G for $i = 1, 2, \ldots, n$.

The Randić index of graph G is defined as the sum of $1/\sqrt{d(v_i)d(v_j)}$ over all edges v_iv_j of G, denoted by $R(G) = \sum_{v_iv_j\in E(G)} 1/\sqrt{d(v_i)d(v_j)}$. This topological index was first proposed by Randić [8] in 1975 under the name "branching index". See the review article [7] for research results on the Randić index.

Let G be a graph of order n with an orientation σ, which assigns to each edge a direction so that G^σ becomes an oriented graph. The graph G is said to be the underlying graph of the oriented graph G^σ. The skew adjacency matrix of G^σ, denoted by $S(G^\sigma)$, is the $n\times n$ real skew-symmetric matrix $(s_{ij})_{n\times n}$ where $s_{ij} = 1$ and $s_{ji} = -1$ if $v_i \to v_j$ is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. The skew energy of an oriented graph G^σ, proposed first in [1] and denoted by $\mathcal{E}_S(G^\sigma)$, is defined as the sum of the norms of all the eigenvalues of $S(G^\sigma)$. Skew energy is an active research area with numerous results on the topic. Adiga et al. [1] showed that the skew

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energy of a directed tree is independent of its orientation, and interestingly it is equal to the energy of the underlying undirected tree. Hou et al. [5] determined the oriented unicyclic graphs with minimal and maximal skew energy. Shen et al. [9] determined the bicyclic digraphs with minimal and maximal skew energy. Gong et al. [2] determined all oriented graphs with minimal skew energy among all connected oriented graphs on n vertices with $m(n \leq m < 2(n-2))$ arcs. For more results on the skew energy of oriented graphs, see [6, 10, 11].

Recently, Gu, Huang, and Li in [3] defined a weighted skew adjacency matrix with Randić weight, the skew Randić matrix, denoted by $R_S(G^\sigma)$, of G^σ as the $n \times n$ real skew-symmetric matrix $(r_{ij})_{n\times n}$, where $r_{ij} = 1/\sqrt{d(v_i)d(v_j)}$ and $r_{ji} = -1/\sqrt{d(v_i)d(v_j)}$ if $v_i \to v_j$ is an arc of G^σ, otherwise $r_{ij} = r_{ji} = 0$. It is obvious that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $R_S(G^\sigma)$ are all purely imaginary numbers. The skew Randić energy of G^σ, denoted by $\mathcal{RE}_S(G^\sigma)$, is defined as the sum of the norms of all the eigenvalues of $R_S(G^\sigma)$, that is, $\mathcal{RE}_S(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|$. In [3], some upper and lower bounds on skew Randić energy are obtained, and it is also proved that the skew Randić energy of an oriented tree is independent of its orientation, and so the skew Randić energy of an oriented tree is the same as the Randić energy of its underlying tree.

Motivated by [3], in this paper, we study the oriented unicyclic graphs with minimal skew Randić energy. Recall that a unicyclic graph is a connected graph with the same number of vertices and edges.

Let $G(n,t)$ be the set of all unicyclic graphs of order n with girth t. Denote the star and the cycle of order n by S_n and C_n, respectively. Let S_n^t be the unicyclic graph of order n obtained by connecting $n-t$ pendant vertices to a vertex of C_t. Figure 1.1 gives the unicyclic graph S_n^3 and its two orientations.

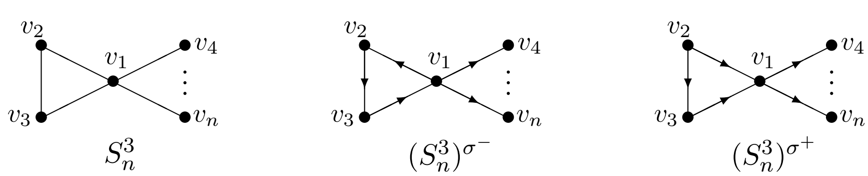

FIGURE 1.1 Unicyclic graph S_n^3 and its two orientations

This paper is organized as follows. In Section 2, we introduce the skew Randić characteristic polynomial of an oriented graph. In Section 3, we present the integral formula for skew Randić energy of an oriented graph. In Section 4, we give some lemmas. In Section 5, the oriented unicyclic graphs with minimal skew Randić energy are determined.

2. The skew Randić characteristic polynomial of an oriented graph

Let G be a graph of order n, and k be a positive integer. A k-matching M of G is a set of k edges such that every vertex of G is incident with at most one edge in M. Denote by $\mathcal{M}_k(G)$ the set of all k-matchings of G, and $m(G,k)$ the number of all k-matchings of G, that is, $m(G,k) = |\mathcal{M}_k(G)|$. In particular, $\mathcal{M}_1(G) = E(G)$, and $m(G,1) = |E(G)|$. A linear subgraph L of G is a disjoint union of some edges and some cycles in G.

Now consider the oriented graph G^σ, and let C be an even cycle of G. We say C is evenly oriented relative to G^σ if it has an even number of edges oriented in the direction of the routing. Otherwise C is oddly oriented. Recall that a linear subgraph L of G is called evenly linear if L contains no odd cycle, and denote by $\mathcal{EL}_i(G)$ the set of all evenly linear subgraphs of G with i vertices. For a linear subgraph $L \in \mathcal{EL}_i(G)$, we denote by $p_e(L)$ and $p_o(L)$ the numbers of evenly oriented cycles and oddly oriented cycles in L relative to G^σ, respectively. If $e = v_iv_j$ is an edge of G, then the Randić value of the edge e is defined as

$$w_R(e) = \left(\frac{1}{\sqrt{d(v_i)d(v_j)}}\right)^2 = \frac{1}{d(v_i)d(v_j)}.$$

If C is a cycle of G, the Randić value of the cycle C is defined as $w_R(C) = 1/(\prod_{v\in V(C)} d(v))$. In general, the Randić value of a linear subgraph L of G, denoted by $w_R(L)$, is defined as the product of Randić values of all edges and cycles in L.

Lemma 2.1 ([3])**.** *Let G be a graph of order n, G^σ be an oriented graph of G with the skew Randić matrix $R_S(G^\sigma)$, and let the skew Randić characteristic polynomial of G^σ be*

$$\phi_{Rs}(G^\sigma, x) = \det(xI - R_S(G^\sigma)) = \sum_{i=0}^{n} b_i(G^\sigma)x^{n-i}. \tag{2.1}$$

Then

$$b_i(G^\sigma) = \sum_{L\in\mathcal{EL}_i(G)} (-2)^{p_e(L)} 2^{p_o(L)} w_R(L). \tag{2.2}$$

Note that the determinant of every real skew-symmetric matrix is non-negative and is 0 if its order is odd. Then we have

(i) $b_0(G^\sigma) = 1$.
(ii) $b_i(G^\sigma) = 0$ for all odd i.
(iii) $b_i(G^\sigma) \geq 0$ for all i.

From Lemma 2.1, we have the following results.

Theorem 2.2. *Let G be a graph of order n and G^σ be an oriented graph of G. Then the skew Randić characteristic polynomial of G^σ can be written* *as*

$$\phi_{Rs}(G^\sigma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i}(G^\sigma) x^{n-2i}, \tag{2.3}$$

where $b_0(G^\sigma) = 1$, *and* $b_{2i}(G^\sigma) \geq 0$ *for* $1 \leq i \leq \lfloor n/2 \rfloor$.

Theorem 2.3. *Let* $G \in G(n,t)$ *be a unicyclic graph with unique cycle* C, G^σ *be an oriented graph of* G, *and the skew Randić characteristic polynomial of* G^σ *be given as in* (2.3). *Then for all* $1 \leq i \leq \lfloor n/2 \rfloor$,

$$b_{2i}(G^\sigma) = \begin{cases} \sum\limits_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha), & \text{if } C \text{ is an odd cycle, or } i < \lfloor t/2 \rfloor, \\ \sum\limits_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) - 2w_R(C) \sum\limits_{\alpha \in \mathcal{M}_{i-\frac{t}{2}}(G-C)} w_R(\alpha), & \\ \qquad \text{if } i \geq \lfloor t/2 \rfloor \text{ and } C \text{ is evenly oriented}, & \\ \sum\limits_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) + 2w_R(C) \sum\limits_{\alpha \in \mathcal{M}_{i-\frac{t}{2}}(G-C)} w_R(\alpha), & \\ \qquad \text{if } i \geq \lfloor t/2 \rfloor \text{ and } C \text{ is oddly oriented}. & \end{cases}$$

3. The integral formula for skew Randić energy of an oriented graph

We first present an integral formula for the skew energy [1] which enables one to compute the skew energy of an oriented graph and compare the skew energy between two oriented graphs without finding the eigenvalues.

Theorem 3.1 ([1]). *Let* $\phi_s(G^\sigma, x) = \det(xI - S(G^\sigma))$ *be the skew characteristic polynomial of an oriented graph* G^σ *on* n *vertices. Then*

$$\mathcal{E}_S(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \Big[n + x \frac{\phi_s'(G^\sigma, -x)}{\phi_s(G^\sigma, -x)} \Big] dx,$$

where $\phi_s'(G^\sigma, x)$ *is the derivative of* $\phi_s(G^\sigma, x)$.

Theorem 3.2 ([5]). *Let* G *be a graph* G *of order* n, *and* G^σ *be an oriented graph of* G *with the skew characteristic polynomial* $\phi_s(G^\sigma, x) = \sum_{i=0}^{n} c_i x^{n-i}$. *Then*

$$\mathcal{E}_S(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \Big[1 + \sum_{i=1}^{\lfloor n/2 \rfloor} c_{2i} x^{2i} \Big] dx.$$

Similar to the results above on the skew energy, we have the following two theorems on the skew Randić energy. We omit their proofs.

Theorem 3.3. *Let* $\phi_{Rs}(G^\sigma, x) = \det(xI - R_S(G^\sigma))$ *be the skew Randić characteristic polynomial of an oriented graph* G^σ *on* n *vertices. Then*

$$\mathcal{RE}_S(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \Big[n + x \frac{\phi_{Rs}'(G^\sigma, -x)}{\phi_{Rs}(G^\sigma, -x)} \Big] dx, \tag{3.1}$$

where $\phi_{Rs}'(G^\sigma, x)$ *is the derivative of* $\phi_{Rs}(G^\sigma, x)$.

Theorem 3.4. *Let G be a graph G of order n, and G^σ be an oriented graph of G with the skew Randić characteristic polynomial $\phi_{Rs}(G^\sigma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i}(G^\sigma)x^{n-2i}$. Then*

$$\mathcal{RE}_S(G^\sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2} \ln\Big[1 + \sum_{i=1}^{\lfloor n/2 \rfloor} b_{2i}(G^\sigma)x^{2i}\Big]dx. \tag{3.2}$$

We call (3.2) the integral formula for the skew Randić energy of an oriented graph. From this integral formula (3.2), it is not difficult to find that for any oriented graph G^σ, $\mathcal{RE}_S(G^\sigma)$ is a strictly monotonically increasing function of these coefficients $b_{2i}(G^\sigma)$ $(i = 1, 2, \ldots, \lfloor n/2 \rfloor)$. Therefore, the method of the quasiorder relation $\preceq$, defined by Gutman and Polansky [4] on graph energy, can be generalized to the skew Randić energy of oriented graphs. To be specific, let

$$\phi_{Rs}(G_1^\sigma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i}(G_1^\sigma)x^{n-2i} \text{ and } \phi_{Rs}(G_2^\tau, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i}(G_2^\tau)x^{n-2i}$$

be skew Randić characteristic polynomials of two oriented graphs G_1^σ and G_2^τ with n vertices, respectively. If $b_{2i}(G_1^\sigma) \geq b_{2i}(G_2^\tau)$ for all $1 \leq i \leq \lfloor n/2 \rfloor$, then set $G_1^\sigma \succeq_R G_2^\tau$ or $G_2^\tau \preceq_R G_1^\sigma$, which implies that $\mathcal{RE}_S(G_1^\sigma) \geq \mathcal{RE}_S(G_2^\tau)$. If $G_1^\sigma \succeq_R G_2^\tau$ and there exists at least one j such that $b_{2j}(G_1^\sigma) > b_{2j}(G_2^\tau)$, then set $G_1^\sigma \succ_R G_2^\tau$, which implies that $\mathcal{RE}_S(G_1^\sigma) > \mathcal{RE}_S(G_2^\tau)$. This quasiordering provides an important method in comparing the skew Randić energies of two oriented graphs.

4. Some lemmas

Let G^σ be an orientation of a graph G, and W a subset of $V(G)$ and $\overline{W} = V(G) \setminus W$. The orientation G^τ of G obtained from G^σ by reversing the direction of all arcs between $\overline{W}$ and W is said to be obtained from G^σ by switching with respect to W. Moreover, two orientations G^σ and G^τ of a graph G are said to be switching-equivalent if G^τ can be obtained from G^σ by a sequence of switching.

Theorem 4.1 ([3]). *Let G^σ and G^τ be two orientations of a graph G. If G^σ and G^τ are switching-equivalent, then $\mathcal{RE}_S(G^\sigma) = \mathcal{RE}_S(G^\tau)$.*

Let $G \in G(n, t)$ be a unicyclic graph with unique cycle C. By Theorem 4.1 and switching-equivalence, there are only two different orientations on G. All edges on the cycle C have the same direction or just one edge on the cycle C has the opposite direction to the directions of other edges on the cycle C regardless of how the edges not on the cycle C are oriented. Denote by G^{σ^-} (G^{σ^+}, respectively) the orientation of G in the first (second, respectively) case above.

Theorem 4.2. *Let $G \in G(n,t)$ be a unicyclic graph with unique cycle C. Then $G^{\sigma^+} \succeq_R G^{\sigma^-}$, and $\mathcal{RE}_S(G^{\sigma^+}) \geq \mathcal{RE}_S(G^{\sigma^-})$. Moreover, if t is odd then $\mathcal{RE}_S(G^{\sigma^+}) = \mathcal{RE}_S(G^{\sigma^-})$; and if t is even then $\mathcal{RE}_S(G^{\sigma^+}) > \mathcal{RE}_S(G^{\sigma^-})$.*

Proof. By Theorem 2.3, if t is odd, then for all $0 \leq i \leq \lfloor n/2 \rfloor$,

$$b_{2i}(G^{\sigma^+}) = b_{2i}(G^{\sigma^-}) = \sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha).$$

If t is even, then for all $0 \leq i < \lfloor t/2 \rfloor$,

$$b_{2i}(G^{\sigma^+}) = b_{2i}(G^{\sigma^-}) = \sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha),$$

and for all $\lfloor t/2 \rfloor \leq i \leq \lfloor n/2 \rfloor$,

$$b_{2i}(G^{\sigma^+}) = \sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) + 2w_R(C) \sum_{\alpha \in \mathcal{M}_{i-\frac{t}{2}}(G-C)} w_R(\alpha),$$

$$b_{2i}(G^{\sigma^-}) = \sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) - 2w_R(C) \sum_{\alpha \in \mathcal{M}_{i-\frac{t}{2}}(G-C)} w_R(\alpha).$$

Thus the theorem holds. □

Theorem 4.3. *For $n \geq 5$, $\mathcal{RE}_S(C_n^\sigma) > \mathcal{RE}_S((S_n^3)^\sigma)$, and for $n = 4$, $\mathcal{RE}_S(C_4^{\sigma^+}) > \mathcal{RE}_S((S_4^3)^\sigma) > \mathcal{RE}_S(C_4^{\sigma^-})$.*

Proof. Note that for $n \geq 5$,

$$b_2((S_n^3)^\sigma) = \sum_{e \in E(S_n^3)} w_R(e) = \frac{n-3}{n-1} + \frac{2}{2(n-1)} + \frac{1}{4} = \frac{5n-9}{4(n-1)},$$

$$b_4((S_n^3)^\sigma) = \sum_{\alpha \in \mathcal{M}_2(S_n^3)} w_R(\alpha) = w_R(v_2v_3) \sum_{k=4}^{n} w_R(v_1v_k) = \frac{n-3}{4(n-1)},$$

$$b_{2i}((S_n^3)^\sigma) = 0 \quad \text{for } i \geq 3,$$

$$b_2(C_n^\sigma) = \sum_{e \in E(C_n)} w_R(e) = \frac{n}{4},$$

$$b_4(C_n^\sigma) = \sum_{\alpha \in \mathcal{M}_2(C_n)} w_R(\alpha) = \frac{1}{16} m(C_n, 2) = \frac{n(n-3)}{32},$$

$$b_{2i}(C_n^\sigma) \geq 0 \quad \text{for } i \geq 3.$$

Then $C_n^\sigma \succ_R (S_n^3)^\sigma$, and $\mathcal{RE}_S(C_n^\sigma) > \mathcal{RE}_S((S_n^3)^\sigma)$ for $n \geq 5$.

For $n = 4$, by direct calculation, we get

$$\mathcal{RE}_S(C_4^{\sigma^-}) = 2, \quad \mathcal{RE}_S(C_4^{\sigma^+}) = 2\sqrt{2},$$

$$\mathcal{RE}_S((S_4^3)^\sigma) = \sqrt{\frac{1}{6}\left(11 - \sqrt{73}\right)} + \sqrt{\frac{1}{6}\left(11 + \sqrt{73}\right)}.$$

So
$$\mathcal{RE}_S(C_4^{\sigma^+}) > \mathcal{RE}_S((S_4^3)^\sigma) > \mathcal{RE}_S(C_4^{\sigma^-}).$$
Then the theorem follows. □

Lemma 4.4. *Let G and H be two graphs of order n as depicted in Figure 4.1, where $d_G(v_1) \geq 3$, $m \geq 3$, and $G_1 = H_1$ is a subgraph of G and H with $v_1 \in V(G_1)$ and $v_1 \in V(H_1)$. Then for $1 \leq i \leq \lfloor n/2 \rfloor$,*
$$\sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{\alpha \in \mathcal{M}_i(H)} w_R(\alpha). \tag{4.1}$$
In particular, the inequality must be strict for $i = 1$.

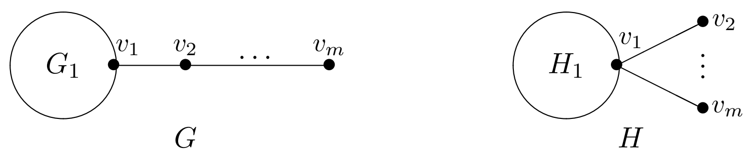

FIGURE 4.1 Graphs G and H

Proof. Set $d_G(v_1) = x$, $G_2 = G-(V(G_1) \setminus \{v_1\})$, and $H_2 = H-(V(H_1) \setminus \{v_1\})$. Then $d_H(v_1) = x + m - 2$, $x \geq 3$, and $m \geq 3$. So for $1 \leq i \leq \lfloor n/2 \rfloor$,
$$\sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{e \in E(G_2)} w_R(e) \sum_{\alpha \in \mathcal{M}_{i-1}(G_1 - v_1)} w_R(\alpha) + \sum_{\alpha \in \mathcal{M}_i(G_1)} w_R(\alpha),$$
and
$$\sum_{\alpha \in \mathcal{M}_i(H)} w_R(\alpha) = \sum_{e \in E(H_2)} w_R(e) \sum_{\alpha \in \mathcal{M}_{i-1}(H_1 - v_1)} w_R(\alpha) + \sum_{\alpha \in \mathcal{M}_i(H_1)} w_R(\alpha).$$
Note that $d_G(v_1) < d_H(v_1)$. Then for $1 \leq i \leq \lfloor n/2 \rfloor$,
$$\sum_{\alpha \in \mathcal{M}_{i-1}(G_1 - v_1)} w_R(\alpha) = \sum_{\alpha \in \mathcal{M}_{i-1}(H_1 - v_1)} w_R(\alpha),$$
$$\sum_{\alpha \in \mathcal{M}_i(G_1)} w_R(\alpha) \geq \sum_{\alpha \in \mathcal{M}_i(H_1)} w_R(\alpha),$$
and
$$\begin{aligned}\sum_{e \in E(G_2)} w_R(e) - \sum_{e \in E(H_2)} w_R(e) &= \frac{1}{2x} + \frac{m-3}{4} + \frac{1}{2} - \frac{m-1}{x+m-2} \\ &= \frac{(m-1)x(x+m-6) + 2(x+m-2)}{4x(x+m-2)} \\ &\geq \frac{2x(x-3) + 2(x+1)}{4x(x+m-2)} \\ &= \frac{(x-1)^2}{2x(x+m-2)} > 0.\end{aligned}$$

That is,

$$\sum_{e\in E(G_2)} w_R(e) > \sum_{e\in E(H_2)} w_R(e).$$

Thus (4.1) holds.

In particular, for $i = 1$,

$$\sum_{\alpha\in\mathcal{M}_1(G)} w_R(e) = \sum_{e\in E(G_1)} w_R(e) + \sum_{e\in E(G_2)} w_R(e)$$
$$> \sum_{e\in E(H_1)} w_R(e) + \sum_{e\in E(H_2)} w_R(e) = \sum_{\alpha\in\mathcal{M}_1(H)} w_R(e).$$

The lemma holds. □

Lemma 4.5. *Let G and H be two graphs of order n as depicted in Figure 4.2, where $d_G(u) \geq 2$, $m \geq 2$, and $G_1 = H_1$ is a subgraph of G and H with $u \in V(G_1)$ and $u \in V(H_1)$. Then for $1 \leq i \leq \lfloor n/2 \rfloor$,*

$$\sum_{\alpha\in\mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{\alpha\in\mathcal{M}_i(H)} w_R(\alpha). \tag{4.2}$$

In particular, the inequality must be strict for $i = 1$.

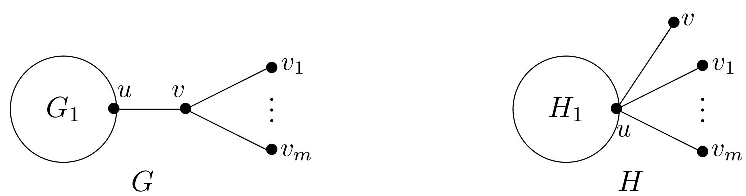

FIGURE 4.2 Graphs G and H

Proof. Set $d_G(u) = x$, $G_2 = G-(V(G_1) \setminus \{u\})$, and $H_2 = H-(V(H_1) \setminus \{u\})$. Then $d_H(u) = x + m$, $x \geq 2$ and $m \geq 2$. So for $1 \leq i \leq \lfloor n/2 \rfloor$,

$$\sum_{\alpha\in\mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{e\in E(G_2)} w_R(e) \sum_{\alpha\in\mathcal{M}_{i-1}(G_1-u)} w_R(\alpha) + \sum_{\alpha\in\mathcal{M}_i(G_1)} w_R(\alpha),$$

and

$$\sum_{\alpha\in\mathcal{M}_i(H)} w_R(\alpha) = \sum_{e\in E(H_2)} w_R(e) \sum_{\alpha\in\mathcal{M}_{i-1}(H_1-u)} w_R(\alpha) + \sum_{\alpha\in\mathcal{M}_i(H_1)} w_R(\alpha).$$

Note that $d_G(u) < d_H(u)$. Then for $1 \leq i \leq \lfloor n/2 \rfloor$,

$$\sum_{\alpha\in\mathcal{M}_{i-1}(G_1-u)} w_R(\alpha) = \sum_{\alpha\in\mathcal{M}_{i-1}(H_1-u)} w_R(\alpha),$$

$$\sum_{\alpha\in\mathcal{M}_i(G_1)} w_R(\alpha) \geq \sum_{\alpha\in\mathcal{M}_i(H_1)} w_R(\alpha),$$

and

$$\begin{aligned}\sum_{e\in E(G_2)} w_R(e) - \sum_{e\in E(H_2)} w_R(e) &= \frac{1}{x(m+1)} + \frac{m}{m+1} - \frac{m+1}{x+m}\\ &= \frac{x+m+mx(x+m)-(m+1)^2x}{x(m+1)(x+m)}\\ &= \frac{m(x-1)^2}{x(m+1)(x+m)} > 0,\end{aligned}$$

that is,

$$\sum_{e\in E(G_2)} w_R(e) > \sum_{e\in E(H_2)} w_R(e).$$

Thus (4.2) holds.

In particular, for $i=1$,

$$\begin{aligned}\sum_{\alpha\in\mathcal{M}_1(G)} w_R(e) &= \sum_{e\in E(G_1)} w_R(e) + \sum_{e\in E(G_2)} w_R(e)\\ &> \sum_{e\in E(H_1)} w_R(e) + \sum_{e\in E(H_2)} w_R(e) = \sum_{\alpha\in\mathcal{M}_1(H)} w_R(e).\end{aligned}$$

The lemma holds. □

From Lemmas 4.4 and 4.5, the following result is clear.

Lemma 4.6. *Let G and H be two graphs of order n as depicted in Figure 4.3, where $d_G(v)\geq 3$, T_m is a tree of order $m\geq 2$ with $v_1\in V(T_m)$, and $G_1=H_1$ is a subgraph of G and H with $v\in V(G_1)$ and $v\in V(H_1)$. Then for $1\leq i\leq \lfloor n/2\rfloor$,*

$$\sum_{\alpha\in\mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{\alpha\in\mathcal{M}_i(H)} w_R(\alpha). \tag{4.3}$$

In particular, the inequality must be strict for $i=1$.

Figure 4.3 Graphs G and H

5. Oriented unicyclic graphs with minimal skew Randić energy

Let $n\geq 5$, $3\leq t\leq n$, $G\in G(n,t)$ and G^σ be an orientation of G. In this section, we give a sharp lower bound for the skew Randić energy of G^σ, and characterize the extremal oriented unicyclic graphs with minimal skew Randić energy.

Lemma 5.1. *Let $t \geq 3$, $m \geq 2$, $n \geq m+t$, G and H be two unicyclic graphs on $G(n,t)$ as depicted in Figure 5.1, where $G_1 = H_1 \in G(n-m,t)$ is a subgraph of G and H with v a vertex on the cycle of G_1 and H_1, and T_m is a tree of order m with $v_1 \in V(T_m)$. If all edges on the unique cycle in G^σ and H^σ have the same direction, then $\mathcal{RE}_S(G^\sigma) > \mathcal{RE}_S(H^\sigma)$.*

FIGURE 5.1 Unicyclic graphs G and H

Proof. By Theorem 2.3 and Lemma 4.6, for all $1 \leq i \leq \lfloor n/2 \rfloor$,

$$\sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{\alpha \in \mathcal{M}_i(H)} w_R(\alpha),$$

and so if t is odd, or $i < \lfloor t/2 \rfloor$, then

$$b_{2i}(G^\sigma) = \sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) \geq \sum_{\alpha \in \mathcal{M}_i(H)} w_R(\alpha) = b_{2i}(H^\sigma),$$

and the inequality is strict for $i = 1$.

In the following, we let t be even and $i \geq (t/2)+1$. Suppose that C_t and C'_t are the cycles of unicyclic graphs G and H, respectively. Since that all edges on the cycle have the same direction, that is, C_t and C'_t are evenly oriented, by Theorem 2.3, we have

$$b_{2i}(G^\sigma) = \sum_{\alpha \in \mathcal{M}_i(G)} w_R(\alpha) - 2w_R(C_t) \sum_{\alpha \in \mathcal{M}_{i-\frac{t}{2}}(G-C_1)} w_R(\alpha),$$

$$b_{2i}(H^\sigma) = \sum_{\alpha \in \mathcal{M}_i(H)} w_R(\alpha) - 2w_R(C'_t) \sum_{\alpha \in \mathcal{M}_{i-\frac{t}{2}}(H-C'_1)} w_R(\alpha).$$

Denote by $\mathcal{M}_i^{(1)}(G)$ the set of all i-matchings of G which contains an $t/2$-matching of C_t, $\mathcal{M}_i^{(1)}(H)$ the set of all i-matchings of H which contains an $t/2$-matching of C'_t, $\mathcal{M}_i^{(2)}(G) = \mathcal{M}_i(G) \setminus \mathcal{M}_i^{(1)}(G)$, and $\mathcal{M}_i^{(2)}(H) = \mathcal{M}_i(H) \setminus \mathcal{M}_i^{(1)}(H)$.

Note that $m(C_t, t/2) = m(C'_t, t/2) = 2$, and for any $\alpha \in \mathcal{M}_{t/2}(C_t)$ and $\beta \in \mathcal{M}_{t/2}(C'_t)$,

$$w_R(\alpha) = \frac{1}{\prod\limits_{w \in V(C_t)} d_G(w)} = w_R(C_t), \quad w_R(\beta) = \frac{1}{\prod\limits_{w \in V(C'_t)} d_H(w)} = w_R(C'_t).$$

Then

$$\begin{aligned}
b_{2i}(G^\sigma) &= \sum_{\alpha\in\mathcal{M}_i(G)} w_R(\alpha) - 2w_R(C_t)\sum_{\alpha\in\mathcal{M}_{i-\frac{t}{2}}(G-C_1)} w_R(\alpha)\\
&= \sum_{\alpha\in\mathcal{M}_i^{(1)}(G)} w_R(\alpha) + \sum_{\alpha\in\mathcal{M}_i^{(2)}(G)} w_R(\alpha) - 2w_R(C_t)\sum_{\alpha\in\mathcal{M}_{i-\frac{t}{2}}(G-C_1)} w_R(\alpha)\\
&= \sum_{\alpha\in\mathcal{M}_{\frac{t}{2}}(C_t)} w_R(\alpha)\sum_{\alpha\in\mathcal{M}_{i-\frac{t}{2}}(G-C_1)} w_R(\alpha) + \sum_{\alpha\in\mathcal{M}_i^{(2)}(G)} w_R(\alpha)\\
&\quad - 2w_R(C_t)\sum_{\alpha\in\mathcal{M}_{i-\frac{t}{2}}(G-C_1)} w_R(\alpha)\\
&= \sum_{\alpha\in\mathcal{M}_i^{(2)}(G)} w_R(\alpha).
\end{aligned}$$

Similar, we have

$$b_{2i}(H^\sigma) = \sum_{\alpha\in\mathcal{M}_i^{(2)}(H)} w_R(\alpha).$$

Note that $3 \le d_G(v) < d_H(v)$. Similar to the proofs of Lemmas 4.4, 4.5, and 4.6, it is easy to see that

$$b_{2i}(G^\sigma) = \sum_{\alpha\in\mathcal{M}_i^{(2)}(G)} w_R(\alpha) \ge \sum_{\alpha\in\mathcal{M}_i^{(2)}(H)} w_R(\alpha) = b_{2i}(H^\sigma).$$

So far, we have $G^\sigma \succ_R H^\sigma$, and so $\mathcal{RE}_S(G^\sigma) > \mathcal{RE}_S(H^\sigma)$. This completes the proof. □

Lemma 5.2. *Let $n \ge 5$, $3 \le t < n$, C_t be a cycle of length t, and $G \in G(n,t)$ $(G \ne S_n^3)$ be a unicyclic graph obtained from the cycle C_t by connecting $n-t$ pendant vertices to some vertices of C_t. Suppose that G^σ is an orientation of G, with all edges on the cycle C_t have the same direction. Then $\mathcal{RE}_S(G^\sigma) > \mathcal{RE}_S((S_n^3)^\sigma)$, where S_n^3 is depicted in Figure 1.1.*

Proof. From the proof of Theorem 4.3,

$$b_2((S_n^3)^\sigma) = \frac{5n-9}{4(n-1)},\quad b_4((S_n^3)^\sigma) = \frac{n-3}{4(n-1)},\quad b_{2i}((S_n^3)^\sigma) = 0 \ \text{ for } i \ge 3.$$

By Theorem 2.2 and the definition of quasiorder relation, in order to prove the theorem, it is sufficient that we prove that $b_2(G^\sigma) > b_2((S_n^3)^\sigma) = \frac{5n-9}{4(n-1)}$, and $b_4(G^\sigma) > b_4((S_n^3)^\sigma) = \frac{n-3}{4(n-1)}$.

First, we prove $b_4(G^\sigma) > b_4((S_n^3)^\sigma)$. Consider the following three cases. *Case 1.1:* C_t has at least three vertices whose degrees are greater than 2 (see Figure 5.2).

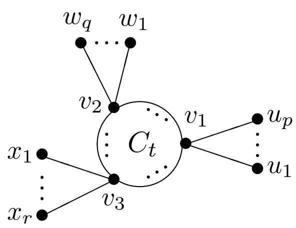

FIGURE 5.2 Graph G for Case 1.1

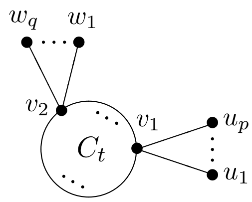

FIGURE 5.3 Graph G for Case 1.2

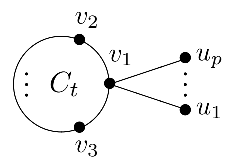

FIGURE 5.4 Graph G for Case 1.3

Let $v_1, v_2, v_3 \in V(C_t)$ with $d(v_i) > 2$ for $i = 1, 2, 3$, and let $d(v_1) = p + 2$, $d(v_2) = q + 2$, and $d(v_3) = r + 2$. Then $p \geq 1$, $q \geq 1$, $r \geq 1$, and

$$\begin{aligned}\sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) &> \sum_{k=1}^{p} w_R(v_1 u_k) \sum_{\ell=1}^{q} w_R(v_2 w_\ell) + \sum_{\ell=1}^{q} w_R(v_2 w_\ell) \sum_{j=1}^{r} w_R(v_3 x_j) \\ &\quad + \sum_{j=1}^{r} w_R(v_3 x_j) \sum_{k=1}^{p} w_R(v_1 u_k) \\ &= \frac{pq}{(p+2)(q+2)} + \frac{qr}{(q+2)(r+2)} + \frac{pr}{(p+2)(r+2)}.\end{aligned}$$

Noting that the function $f(x) = x/(x+2)$ is increasing on $x \geq 1$, we have $f(x) \geq f(1) = 1/3$ for $x \geq 1$. This implies that

$$\sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) > \frac{1}{3}.$$

If t is odd or $t > 4$, then

$$b_4(G^\sigma) = \sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) > \frac{1}{3} > \frac{n-3}{4(n-1)}.$$

If $t = 4$, then

$$w_R(C_t) = \frac{1}{\prod_{v \in V(C_t)} d(v)} \leq \frac{1}{(q+2)(q+2)(r+2)} \leq \frac{1}{3^3},$$

and so

$$b_4(G^\sigma) = \sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) - 2w_R(C_t) > \frac{1}{3} - \frac{1}{3^3} > \frac{n-3}{4(n-1)}.$$

Thus $b_4(G^\sigma) > b_4((S_n^3)^\sigma)$ for Case 1.1.

Case 1.2: C_t has exactly two vertices whose degrees are greater than 2 (see Figure 5.3).

Let $v_1, v_2 \in V(C_t)$ with $d(v_1) > 2$ and $d(v_2) > 2$, and set $d(v_1) = p + 2$ and $d(v_2) = q + 2$. Then $p \geq 1$, $q \geq 1$, and $p + q = n - t$.

If $t \geq 5$, then there is an edge $e \in E(C_t)$ which endpoints have degree 2. Thus

$$\begin{aligned}
\sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) &> \sum_{k=1}^{p} w_R(v_1u_k) \sum_{\ell=1}^{q} w_R(v_2w_\ell) \\
&\quad + w_R(e) \left(\sum_{k=1}^{p} w_R(v_1u_k) + \sum_{\ell=1}^{q} w_R(v_2w_\ell) \right) \\
&= \frac{pq}{(p+2)(q+2)} + \frac{p}{4(p+2)} + \frac{q}{4(q+2)} \\
&\geq \left(\frac{pq}{(p+2)(q+2)} + \frac{p}{4(p+2)} + \frac{q}{4(q+2)} \right) \Big|_{p=q=1} = \frac{5}{18}.
\end{aligned}$$

This implies that

$$b_4(G^\sigma) = \sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) > \frac{5}{18} > \frac{n-3}{4(n-1)}.$$

If $t = 4$, then there are the following two cases as depicted in Figure 5.5.

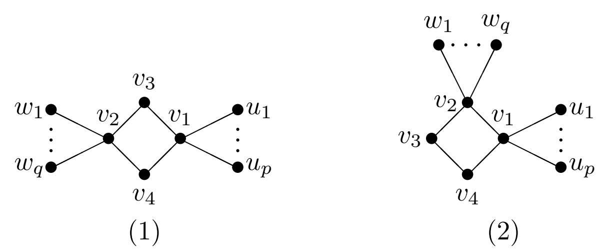

FIGURE 5.5 Graph G for $t = 4$

(1) If v_1 and v_2 are not adjacent (see Figure 5.5(1)), then

$$\begin{aligned}
\sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) &= \sum_{k=1}^{p} w_R(v_1u_k) \sum_{\ell=1}^{q} w_R(v_2w_\ell) \\
&\quad + (w_R(v_1v_3) + w_R(v_1v_4)) \sum_{\ell=1}^{q} w_R(v_2w_\ell) \\
&\quad + (w_R(v_2v_3) + w_R(v_2v_4)) \sum_{k=1}^{p} w_R(v_1u_k) \\
&\quad + w_R(v_1v_3)w_R(v_2v_4) + w_R(v_2v_3)w_R(v_1v_4) \\
&= \frac{pq}{(p+2)(q+2)} + \frac{q}{(p+2)(q+2)} \\
&\quad + \frac{p}{(p+2)(q+2)} + \frac{1}{2(p+2)(q+2)} \\
&= \frac{2pq+2p+2q+1}{2(p+2)(q+2)},
\end{aligned}$$

and

$$w_R(C_t) = \frac{1}{4(p+2)(q+2)}.$$

So

$$b_4(G^\sigma) = \sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) - 2w_R(C_t) = \frac{pq+p+q}{(p+2)(q+2)}$$
$$\geq \frac{pq+p+q}{(p+2)(q+2)}\Big|_{p=1,q=1} = \frac{1}{3} > \frac{n-3}{4(n-1)}.$$

(2) If v_1 and v_2 are adjacent (see Figure 5.5(2)), then

$$\begin{aligned}\sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) &= \sum_{k=1}^{p} w_R(v_1u_k) \sum_{\ell=1}^{q} w_R(v_2w_\ell) \\ &\quad + (w_R(v_3v_4) + w_R(v_1v_4)) \sum_{\ell=1}^{q} w_R(v_2w_\ell) \\ &\quad + (w_R(v_2v_3) + w_R(v_3v_4)) \sum_{k=1}^{p} w_R(v_1u_k) \\ &\quad + w_R(v_1v_2)w_R(v_3v_4) + w_R(v_2v_3)w_R(v_1v_4) \\ &= \frac{pq}{(p+2)(q+2)} + \frac{p+4}{4(p+2)}\frac{q}{q+2} \\ &\quad + \frac{q+4}{4(q+2)}\frac{p}{p+2} + \frac{1}{2(p+2)(q+2)} \\ &= \frac{3pq+2p+2q+1}{2(p+2)(q+2)},\end{aligned}$$

and

$$w_R(C_t) = \frac{1}{4(p+2)(q+2)}.$$

So

$$b_4(G^\sigma) = \sum_{\alpha \in \mathcal{M}_2(G)} w_R(\alpha) - 2w_R(C_t) = \frac{6pq+4p+4q+1}{4(p+2)(q+2)}$$
$$\geq \frac{6pq+4p+4q+1}{4(p+2)(q+2)}\Big|_{p=1,q=1} = \frac{5}{12} > \frac{n-3}{4(n-1)}.$$

If $t = 3$ (see Figure 5.6), then $n = p + q + 3$, and

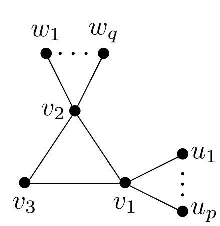

FIGURE 5.6 Graph G for $t = 3$

$$\begin{aligned}\sum_{\alpha\in\mathcal{M}_2(G)} w_R(\alpha) &= \sum_{k=1}^{p} w_R(v_1u_k)\sum_{\ell=1}^{q} w_R(v_2w_\ell) + w_R(v_2v_3)\sum_{k=1}^{p} w_R(v_1u_k)\\ &\quad + w_R(v_1v_3)\sum_{\ell=1}^{q} w_R(v_2w_\ell)\\ &= \frac{pq}{(p+2)(q+2)} + \frac{p}{2(p+2)(q+2)} + \frac{q}{2(p+2)(q+2)}\\ &= \frac{2pq+p+q}{2(pq+2q+2q+4)}\\ &= \frac{2p(n-p-3)+n-3}{2(p(n-p-3)+2(n-3)+4)} = \frac{2np+n-2p^2-6p-3}{2(p+2)(n-p-1)}\\ &\geq \frac{2np+n-2p^2-6p-3}{2(p+2)(n-p-1)}\Big|_{p=1} = \frac{3n-11}{6(n-2)}.\end{aligned}$$

This implies that

$$b_4(G^\sigma) = \sum_{\alpha\in\mathcal{M}_2(G)} w_R(\alpha) \geq \frac{3n-11}{6(n-2)} > \frac{n-3}{4(n-1)}.$$

Thus $b_4(G^\sigma) > b_4((S_n^3)^\sigma)$ for Case 1.2.

Case 1.3: C_t has exactly one vertex whose degree is greater than 2 (see Figure 5.4).

Let $v_1 \in V(C_t)$ with $d(v_1) > 2$, and set $d(v_1) = p+2$. Then $p = n-t$. Since $G \neq S_n^3$, we have $t \geq 4$, and there are $t-2$ edges on C_t which endpoints have degree 2. Then

$$\begin{aligned}\sum_{\alpha\in\mathcal{M}_2(G)} w_R(\alpha) &\geq \sum_{e\in E(C_t-v_1)} w_R(e)\sum_{k=1}^{p} w_R(v_1u_k) + w_R(v_1v_2)\sum_{e\in E(C_t-v_1-v_2)} w_R(e)\\ &\quad + w_R(v_1v_3)\sum_{e\in E(C_t-v_1-v_3)} w_R(e)\\ &= \frac{(t-2)p}{4(p+2)} + \frac{t-3}{8(p+2)} + \frac{t-3}{8(p+2)} = \frac{(t-2)p+(t-3)}{4(p+2)},\end{aligned}$$

and

$$w_R(C_t) = \frac{1}{2^{t-1}(p+2)}.$$

If $t \geq 5$, then

$$b_4(G^\sigma) = \sum_{\alpha\in\mathcal{M}_2(G)} w_R(\alpha) \geq \frac{(t-2)p+(t-3)}{4(p+2)} \geq \frac{5}{12} > \frac{n-3}{4(n-1)}.$$

If $t=4$, then $p=n-4$, and

$$b_4(G^\sigma)=\sum_{\alpha\in\mathcal{M}_2(G)}w_R(\alpha)-2w_R(C_t)\geq\frac{(t-2)p+(t-3)}{4(p+2)}-\frac{1}{2^{t-2}(p+2)}$$
$$=\frac{2n-7}{4(n-2)}-\frac{1}{4(n-2)}=\frac{n-4}{2(n-2)}>\frac{n-3}{4(n-1)}.$$

Thus $b_4(G^\sigma)>b_4((S_n^3)^\sigma)$ for Case 1.3.

Next, we prove $b_2(G^\sigma)>b_2((S_n^3)^\sigma)$. Consider the following four cases.

Subcase 2.1: C_t has at least four vertices whose degrees are greater than 2.

Let $v_i\in V(C_t)$ with $d(v_i)>2$ for $i=1,2,3,4$, and let their degrees be $p+2$, $q+2$, $r+2$, and $s+2$, respectively. Then

$$b_2(G^\sigma)=\sum_{e\in E(G)}w_R(e)\geq\frac{p}{p+2}+\frac{q}{q+2}+\frac{r}{r+2}+\frac{s}{s+2}\geq\frac{4}{3}>\frac{5n-9}{4(n-1)}.$$

Subcase 2.2: C_t has exactly three vertices whose degrees are greater than 2 (see Figure 5.2).

Let $v_i\in V(C_t)$ with $d(v_i)>2$ for $i=1,2,3$, and set $d(v_1)=p+2$, $d(v_2)=q+2$, and $d(v_3)=r+2$. Then

$$\begin{aligned}b_2(G^\sigma)&=\sum_{e\in E(G)}w_R(e)\\&\geq\frac{p}{p+2}+\frac{q}{q+2}+\frac{r}{r+2}+\frac{1}{(p+2)(q+2)}\\&\quad+\frac{1}{(p+2)(r+2)}+\frac{1}{(q+2)(r+2)}\\&=\frac{3pqr+4pq+4pr+4qr+5p+5q+5r+6}{(p+2)(q+2)(r+2)}\\&\geq\left.\frac{3pqr+4pq+4pr+4qr+5p+5q+5r+6}{(p+2)(q+2)(r+2)}\right|_{p=q=r=1}\\&=\frac{4}{3}>\frac{5n-9}{4(n-1)}.\end{aligned}$$

Subcase 2.3: C_t has exactly two vertices whose degrees are greater than 2 (see Figure 5.3).

Let $v_1,v_2\in V(C_t)$ with $d(v_1)>2$ and $d(v_2)>2$, and set $d(v_1)=p+2$ and $d(v_2)=q+2$. Then $p+q=n-t$.

If v_1 and v_2 are adjacent, then

$$\begin{aligned}b_2(G^\sigma)&=\sum_{e\in E(G)}w_R(e)\\&=\frac{p}{p+2}+\frac{q}{q+2}+\frac{1}{2(p+2)}+\frac{1}{2(q+2)}+\frac{1}{(p+2)(q+2)}+\frac{t-3}{4}\\&=\frac{2(p+q)(t+2)+pq(t+5)+4t}{4(p+2)(q+2)}.\end{aligned}$$

If $t \geq 4$, then

$$b_2(G^\sigma) \geq \frac{(3p+4)(3q+4)}{4(p+2)(q+2)} \geq \frac{49}{36} > \frac{5n-9}{4(n-1)}.$$

If $t = 3$, noting that $q = n - p - 3 \geq 1$, then

$$b_2(G^\sigma) = \frac{4np + 5n - 4p^2 - 12p - 9}{2(p+2)(n-p-1)},$$

and

$$\begin{aligned} b_2(G^\sigma) - b_2((S_n^3)^\sigma) &= \frac{4np + 5n - 4p^2 - 12p - 9}{2(p+2)(n-p-1)} - \frac{5n-9}{4(n-1)} \\ &= \frac{p(3n+1)(n-p-3)}{4(n-1)(p+2)(n-p-1)} > 0. \end{aligned}$$

This implies that $b_2(G^\sigma) > b_2((S_n^3)^\sigma)$.

If v_1 and v_2 are not adjacent, then $t \geq 4$,

$$\begin{aligned} b_2(G^\sigma) = \sum_{e \in E(G)} w_R(e) &\geq \frac{p}{p+2} + \frac{q}{q+2} + \frac{2}{2(p+2)} + \frac{2}{2(q+2)} \\ &= \frac{p+1}{p+2} + \frac{q+1}{q+2} \geq \frac{2}{3} + \frac{2}{3} > \frac{5n-9}{4(n-1)}. \end{aligned}$$

Subcase 2.4: C_t has exactly one vertex whose degree is greater than 2 (see Figure 5.4).

Let $v_1 \in V(C_t)$ with $d(v_1) > 2$, and set $d(v_1) = p + 2$. Then $p = n - t$, $t \geq 4$, and

$$\begin{aligned} b_2(G^\sigma) = \sum_{e \in E(G)} w_R(e) &= \frac{p}{p+2} + \frac{1}{2(p+2)} + \frac{1}{2(p+2)} + \frac{t-2}{4} \\ &= \frac{nt + 2n - t^2}{4(n-t+2)} \geq \frac{nt + 2n - t^2}{4(n-t+2)}\Big|_{t=4} = \frac{3n-8}{2(n-2)} > \frac{5n-9}{4(n-1)}. \end{aligned}$$

So far, we have proved that $b_2(G^\sigma) > b_2((S_n^3)^\sigma)$ and $b_4(G^\sigma) > b_4((S_n^3)^\sigma)$. Then $G^\sigma \succ_R (S_n^3)^\sigma$, and $\mathcal{RE}_S(G^\sigma) > \mathcal{RE}_S((S_n^3)^\sigma)$. This completes the proof. □

By Theorems 4.2 and 4.3 and Lemmas 5.1 and 5.2, we obtain the following main theorem.

Theorem 5.3. *Let $n \geq 4$, $3 \leq t \leq n$, $G \in G(n,t)$ be a unicyclic graph, and G^σ be an orientation of G. Then*

(1) for $n \geq 5$,

$$\mathcal{RE}_S(G^\sigma) \geq \mathcal{RE}_S((S_n^3)^\sigma),$$

with equality if and only if G is isomorphic to S_n^3; and

(2) for $n = 4$,

$$\mathcal{RE}_S(G^\sigma) \geq \mathcal{RE}_S(C_4^{\sigma^-}),$$

with equality if and only if G^σ and $C_4^{\sigma^-}$ are switching-equivalent.

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