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HOMOTOPY TYPE OF INDEPENDENCE COMPLEXES OF CERTAIN FAMILIES OF GRAPHS

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Abstract. We show that the independence complexes of the generalised Mycielskian of complete graphs are homotopy equivalent to a wedge sum of spheres and determine the number of copies and the dimensions of these spheres. We also prove that the independence complexes of the categorical product of complete graphs are wedge sum of circles, up to homotopy. Further, we show that if we perturb a graph G in a certain way, then the independence complex of this new graph is homotopy equivalent to the suspension of the independence complex of G.

1. Introduction

A subset I of the vertex set of a graph G is called *independent*, if the induced subgraph of G on I is a collection of isolated vertices. The *independence complex*, $\mathrm{Ind}(G)$, of a simple graph G is the simplicial complex whose simplices are the independent sets of G. In the last few years a lot of attention has been drawn towards the study of independence complexes of graphs.

In [2], Babson and Kozlov used the topology of independence complexes of cycles to prove a conjecture by Lovász. Meshulam, in [15], gave a connection between the domination number of a graph G and certain homological properties of $\mathrm{Ind}(G)$, and their application to Hall-type theorems for coloured independent sets. Properties of independence complexes have also been used to study the Tverberg graphs [8] and the independent system of representatives [1].

Even after two decades of studying these complexes, there are very few classes of graphs for which a closed-form formula for the homotopy type of their independence complexes is known. For instance, see [4] for stable Kneser graphs, [11] for forests, [13] for cycle graphs, and [16] for a family of

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regular bipartite graphs. For more on these complexes, interested readers are referred to [6, 7, 12].

In this article, we do the computation for the homotopy type of independence complexes of certain classes of graphs. We also give a closed-form formula for some of these classes. This article is arranged as follows. In Section 3, we analyse the independence complexes of the product of complete graphs and show that it is homotopy equivalent to a wedge of circles (cf. Proposition 3.4). Section 4 is devoted towards the computation of independence complexes of the generalised Mycielskian (see Definition 2.2) of complete graphs (cf. Theorem 4.14), which turns out to be a wedge of spheres. In Section 5, we show that if we perturb a graph G in a particular manner (by removing some edges and adding new edges and vertices; see Figure 4) to obtain a new graph H, then $\mathrm{Ind}(H)$ is homotopy equivalent to the suspension of $\mathrm{Ind}(G)$ (cf. Theorem 5.1).

As an application of Theorem 5.1, we determine the homotopy type of independence complexes of cycles with a particular type of subdivision.

2. Preliminaries

In this section, we recall various standard definitions, notations, and results which will be used in this article.

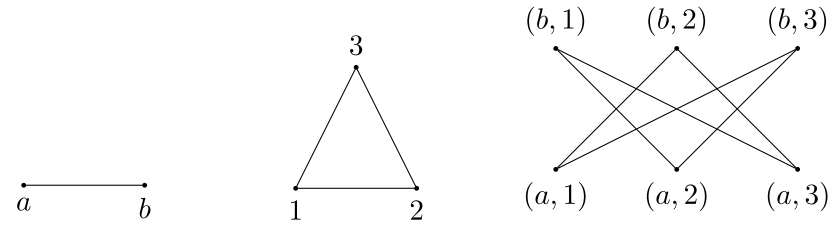
A *graph* is an ordered pair $G = (V, E)$ where V is called the set of vertices and $E \subseteq V \times V$, the set of unordered edges of G. The vertices $v_1, v_2 \in V$ are said to be adjacent, if $(v_1, v_2) \in E$. This is also denoted by $v_1 \sim v_2$, and if $v_1 = v_2$, then v_1 is said to be a *looped* vertex. For a vertex v of G, the set of its *neighbours* in G is $\{x \in V(G) : x \sim v\}$, and is denoted by $N(v)$. We fix the notation $N[v]$ to denote the set $N(v) \cup \{v\}$. Also, if $A \subseteq V(G)$, then $N(A) := \bigcup_{v \in A} N(v)$ and $N[A] := \bigcup_{v \in A} N[v]$.

A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of the graph G. For a non-empty subset U of $V(G)$, the induced subgraph $G[U]$, is the subgraph of G with vertices $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) \mid a, b \in U\}$. In this article, $G[V(G) \setminus A]$ will be denoted by $G - A$ for $A \subsetneq V(G)$.

The *complete graph* on n vertices is a graph where any two distinct vertices are adjacent, and it is denoted by K_n. For $n \geq 3$, the *cycle graph* C_n is the graph with $V(C_n) = \{1, \ldots, n\}$ and $E(C_n) = \{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(1, n)\}$.

Definition 2.1. *The categorical product of two graphs G and H, denoted by $G \times H$ is the graph where $V(G \times H) = V(G) \times V(H)$ and $(g, h) \sim (g', h')$ in $G \times H$, if and only if $g \sim g'$ in G and $h \sim h'$ in H. For an example, see Figure 1.*

For $r \geq 1$, let L_r denote the path graph of length r with loop at one end, i.e., it is a graph with vertex set $V(L_r) = \{0, \ldots, r\}$ and edge set $E(L_r) = \{(i, i+1) \mid 0 \leq i \leq r-1\} \cup \{(0, 0)\}$.



(A) K_2 (B) K_3 (C) $K_2 \times K_3$

FIGURE 1. Categorical product of K_2 and K_3

Definition 2.2. *Let G be a graph and $r \geq 1$. The r-th generalised Mycielskian, $M_r(G)$, of G is the graph $(G \times L_r)/\sim_r$, where $\sim_r$ is the equivalence which identifies all those vertices whose second coordinate is r. The graph $M_2(G)$ is called the Mycielskian of G. See Figure 2 for $M_2(K_3)$, where the vertex set of K_3 is taken to be $\{a, b, c\}$.*

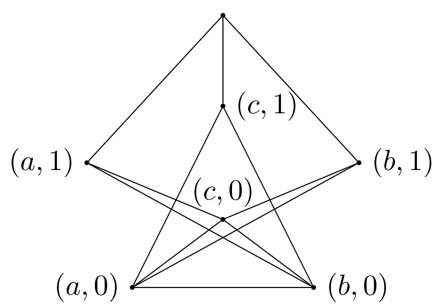


FIGURE 2. $M_2(K_3)$

An *(abstract) simplicial complex* K is a collection of finite sets such that if $\tau \in K$ and $\sigma \subset \tau$, then $\sigma \in K$. The elements of K are called the *simplices* of K. If $\sigma \in K$ and $|\sigma| = k+1$, then σ is said to be *k-dimensional.* The set of 0-dimensional simplices of K is denoted by $V(K)$, and its elements are called *vertices* of K. A *subcomplex* of a simplicial complex K is a simplicial complex whose simplices are contained in K. In this article, we always assume the empty set as a simplex of any simplicial complex.

The link of a vertex $v \in V(K)$ is the subcomplex of K defined as

$$lk_K(v) := \{\sigma \in K \mid v \notin \sigma \text{ and } \sigma \cup \{v\} \in K\}.$$

The *star* of a simplex $\sigma \in K$ is the subcomplex of K defined as

$$st_K(\sigma) := \{\tau \in K \mid \sigma \cup \tau \in K\}.$$

In this article, we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to the book [14] by Kozlov.

Definition 2.3 ([3]). *Let K be a simplicial complex and $\sigma \in K$. The star cluster of σ in K is a subcomplex of K defined as*

$$SC_K(\sigma) := \bigcup_{u \in \sigma} st_K(\{u\}).$$

The following results by Barmak will be used in this article.

Lemma 2.4 ([3, Lemma 3.2]). *The star cluster of a simplex in independence complex is contractible.*

We recall the following definitions from [10, Chapter 0]. For a space X, the *suspension* $\Sigma(X)$ is the quotient of $X \times [0,1]$ obtained by collapsing $X \times 0$ to one point and $X \times 1$ to another point. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the *wedge* of X and Y, denoted $X \bigvee Y$, is the quotient of the disjoint union $X \sqcup Y$ obtained by identifying x_0 and y_0 to a single point.

Lemma 2.5 ([3, Lemma 3.3]). *Let K_1 and K_2 be two contractible subcomplexes of a simplicial complex K such that $K = K_1 \cup K_2$. Then $K \simeq \Sigma(K_1 \cap K_2)$, where $\Sigma(X)$ denotes the suspension of space X.*

Lemma 2.6 ([3, Theorem 3.6]). *Let G be a graph and v be a non-isolated vertex of G which is contained in no triangle. Then $N(v)$ is a simplex of $\mathrm{Ind}(G)$, and*

$$\mathrm{Ind}(G) \simeq \Sigma(st_{\mathrm{Ind}(G)}(\{v\}) \cap SC_{\mathrm{Ind}(G)}\big(N(v)\big)).$$

The following observation directly follows from the definition of independence complexes of graphs.

Lemma 2.7. *Let G be a graph obtained by taking disjoint union of two graphs G_1 and G_2. Then,*

$$\mathrm{Ind}(G) = \mathrm{Ind}(G_1 \sqcup G_2) \simeq \mathrm{Ind}(G_1) * \mathrm{Ind}(G_2),$$

where $*$ denotes the join operation.

Now we discuss some tools needed from discrete Morse theory ([9]).

Definition 2.8 ([14, Definition 11.1]). *A partial matching on a poset P is a subset $M \subseteq P \times P$ such that*

(i) *$(a, b) \in M$ implies $b \succ a$; i.e., $a < b$ and no c satisfies $a < c < b$, and*
(ii) *each $a \in P$ belong to at most one element in M.*

Note that, $\mathcal{M}$ is a partial matching on a poset P if and only if there exists $\mathcal{A} \subset P$ and an injective map $\mu : \mathcal{A} \to P \setminus \mathcal{A}$ such that $\mu(a) \succ a$ for all $a \in \mathcal{A}$.

Definition 2.9. *An acyclic matching is a partial matching $\mathcal{M}$ on the poset P such that there does not exist a cycle*

$$\mu(a_1) \succ a_1 \prec \mu(a_2) \succ a_2 \prec \mu(a_3) \succ a_3 \dots \mu(a_t) \succ a_t \prec \mu(a_1), t \geq 2.$$

For an acyclic partial matching on P, those elements of P which do not belong to the matching are said to be *critical* .

Theorem 2.10 ([14, Theorem 11.13])**.** *Let Δ be a simplicial complex and M be an acyclic matching on the face poset of Δ. Let c_i denote the number of critical i-dimensional cells of Δ with respect to the matching M. Then Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geq 0$, plus a single 0-dimensional cell in the case where the empty set is also paired in the matching.*

The following can be inferred from Theorem 2.10.

Remark 2.11: If an acyclic matching has critical cells only in a fixed dimension i, then Δ is homotopy equivalent to a wedge of i-dimensional spheres.

3. Independence complex of $K_m \times K_n$

In this section, we compute the independence complex of $K_m \times K_n$ for $m, n \geq 2$. We first start by defining an acyclic matching on the face poset of a general simplicial complex; then use a special case of this matching to prove the result for $\text{Ind}(K_m \times K_n)$.

Let K be a simplicial complex and let $X = \{x_1, \ldots, x_n\} \subseteq V(K)$. The elements of X are ordered as; $x_1 < x_2 < \cdots < x_n$.

Let P be the face poset of $(K, \subseteq)$. For $1 \leq i \leq n$, define

$$A_{x_i} = \{\sigma \in A'_{x_{i-1}} \mid x_i \notin \sigma, \text{ and } \sigma \cup \{x_i\} \in A'_{x_{i-1}}\}, \text{ where } A'_{x_0} = P,$$

$$\mu_{x_i} : A_{x_i} \to A'_{x_{i-1}} \setminus A_{x_i} \text{ by } \mu_{x_i}(\sigma) = \sigma \cup \{x_i\} \text{ and}$$

$$A'_{x_i} = A'_{x_{i-1}} \setminus \mathfrak{S}_{x_i}, \text{ where } \mathfrak{S}_{x_i} = A_{x_i} \cup \mu_{x_i}(A_{x_i}).$$

We note that by construction, $A_{x_i} \cap A_{x_j} = \emptyset$ whenever $i \neq j$. Let $A = \bigcup_{i=1}^{n} A_{x_i}$ and $\mu_K^X : A \to P \setminus A$ be defined by $\mu_K^X(\sigma) = \mu_{x_i}(\sigma)$, where x_i is the unique element such that $\sigma \in A_{x_i}$.

Clearly, μ_K^X is injective and is, therefore, a well-defined partial matching on P. The next example illustrates the above construction of matching.

Example 3.1. *Let $K = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}\}$ be a simplicial complex. Let $(P, \subseteq)$ be the face poset of K. Let $X = \{1,2\}$ and define an order on X by $1 < 2$. The construction of the matching μ_K^X is as follows: To start with, $A'_0 = P$, $A_1 = \{\{\emptyset\}, \{2\}, \{3\}, \{2,3\}\}$, then $\mu_1(A_1) = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ and $\mathfrak{S}_1 = A_1 \cup \mu_1(A_1) = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Thus $A'_1 = A'_0 \setminus \mathfrak{S}_1 = P \setminus \mathfrak{S}_1 = \{\{4\}, \{2,4\}, \{3,4\}\}$. Now, $A_2 = \{\{4\}\}$, $\mu_2(A_2) = \{\{2,4\}\}$ and $\mathfrak{S}_2 = \{\{4\}, \{2,4\}\}$. Hence the set of unmatched cells is $A'_2 = A_1 \setminus \mathfrak{S}_2 = \{\{3,4\}\}$.*

It follows from [16, Proposition 3.1] that μ_K^X is an acyclic matching. For the sake of completeness, we give a proof here as well.

Proposition 3.2. *μ_K^X is an acyclic matching on P.*

Proof. Let there exist distinct cells $\sigma_1, \ldots, \sigma_t \in A$ such that $\mu_K^X(\sigma_i) \succ \sigma_{i+1 \pmod t}, 1 \leq i \leq t$.

Let $x \in X$ be the least element such that $\{\sigma_1, \ldots, \sigma_t\} \cap A_x \neq \emptyset$. Without loss of generality, assume that $\sigma_1 \in A_x$, i.e., $x \notin \sigma_1$ and $\mu_K^X(\sigma_1) = \sigma_1 \cup \{x\}$. $\mu_K^X(\sigma_1) \succ \sigma_2$ and $\sigma_1 \neq \sigma_2$ implies that there exists $x' \in \mu_1(\sigma_1), x' \neq x$ such that $\sigma_2 = \mu_K^X(\sigma_1) \setminus \{x'\}$. We now have the following two possibilities:

(1) $x \in \sigma_t$.
  $\sigma_1 \in A_x$ implies that $x \notin \sigma_1$. $x \in \sigma_t$ implies that $x \in \mu_K^X(\sigma_t)$. Therefore, $\sigma_1 = \mu_K^X(\sigma_t) \setminus \{x\}$ which implies that $\mu_K^X(\sigma_1) = \mu(\sigma_t)$ a contradiction, since $\sigma_1 \neq \sigma_t$.

(2) $x \notin \sigma_t$, i.e., there exists a least $l \in \{2, \ldots, t\}$ such that $x \notin \sigma_l$.
  $x \in \mu_K^X(\sigma_{l-1})$ and $x \notin \sigma_l$ implies that $\sigma_l = \mu_K^X(\sigma_{l-1}) \setminus \{x\}$ i.e., $\mu_K^X(\sigma_{l-1}) = \sigma_l \cup \{x\}$. Since σ_l and $\mu_K^X(\sigma_{l-1}) \notin A_i \cup \mu_{x_i}(x_i)\ \forall\ i < x$, from the definition $\sigma_l \in A_x$. This implies that $\mu_K^X(\sigma_l) = \sigma_l \cup \{x\} = \mu_K^X(\sigma_{l-1})$, which implies that $\sigma_l = \sigma_{l-1}$, a contradiction.

Therefore, μ_K^X is an acyclic matching on P. □

Let $m, n \geq 2$ and $V(K_m) = \{a_1, \ldots, a_m\}$, $V(K_n) = \{b_1, \ldots, b_n\}$.

Remark 3.3: Observe that the maximal simplices of $\mathrm{Ind}(K_m \times K_n)$ are only of the following two types:

(1) sets of the form $\{(a_i, b_j) \mid j \in [n]\}$, where $i \in [m]$, and
(2) sets of the form $\{(a_i, b_j) \mid i \in [m]\}$, where $j \in [n]$.

Using the above classification of simplices of $\mathrm{Ind}(K_m \times K_n)$, we prove the following result.

Proposition 3.4. *Let $m, n \geq 2$. Then*

$$\mathrm{Ind}(K_m \times K_n) \simeq \bigvee_{(m-1)(n-1)} \mathbb{S}^1.$$

Proof. Let $I := \mathrm{Ind}(K_m \times K_n)$ and let $J = \{(a_1, b_i) \mid 1 \leq i \leq n\} \cup \{(a_i, b_1) \mid 2 \leq i \leq m\} \subseteq V(I)$. Further, let $P_{m,n}$ be the face poset of $(I, \subseteq)$. We define the ordering on J as follows:

$$(a_1, b_1) < \cdots < (a_1, b_n) < (a_2, b_1) < (a_3, b_1) < \cdots < (a_m, b_1).$$

Let μ_I^J be the matching defined as in the beginning of this section with respect to the ordering of elements of J given as above. From Proposition 3.2, μ_I^J is an acyclic matching. Let C be the set of critical cells for the matching μ_I^J.

Claim 3.5. $C = \{\{(a_i, b_1), (a_i, b_j)\} \mid 2 \leq i \leq m, 2 \leq j \leq n\}$.

Proof of Claim 3.5. In this proof, for the convenience of notation, we denote μ_I^J by μ.

Here, we first show that every element of C is critical. Let $i \in \{2, \ldots, m\}$ and $j \in \{2, \ldots, n\}$. First observe that $\mu(\{(a_i, b_j)\}) = \{(a_1, b_j), (a_i, b_j)\}$.

Since $i, j \geq 2$, it follows from the definition of μ that $\{(a_i, b_1), (a_i, b_j)\}$ is a critical cell.

Now, let $\sigma \in I$ be a critical cell. Note that $\mu(\{\emptyset\}) = \{(a_1, b_1)\}$, therefore $\sigma \neq \{(a_1, b_1)\}$. Since for each $j \geq 2$, $\mu(\{(a_1, b_j)\}) = \{(a_1, b_j), (a_1, b_1)\}$; and for each $i \geq 2$ and $k \geq 1$, $\mu(\{(a_i, b_k)\}) = \{(a_i, b_k), (a_1, b_k)\}$, we thus conclude that σ has at least two elements. From Remark 3.3, we get that either $\sigma = \{(a_{i_1}, b_j), \ldots, (a_{i_t}, b_j)\}$ for some fixed $j \in [n]$ and $t \geq 2$ or $\sigma = \{(a_i, b_{j_1}), \ldots, (a_i, b_{j_l})\}$ for some fixed $i \in [m]$ and $l \geq 2$.

Suppose $\sigma = \{(a_{i_1}, b_j), \ldots, (a_{i_t}, b_j)\}$ for some $j \in [n]$ and $t \geq 2$. If $(a_1, b_j) \notin \sigma$, then $\mu(\sigma) = \sigma \cup \{(a_1, b_j)\}$; and if $(a_1, b_j) \in \sigma$, then $\sigma = \mu(\sigma \setminus \{(a_1, b_j)\})$, which contradicts that σ is a critical cell. Therefore, $\sigma = \{(a_i, b_{j_1}), \ldots, (a_i, b_{j_l})\}$ for some $i \in [m]$ and $l \geq 2$.

Note that, if $(a_i, b_1) \notin \sigma$ then $\mu(\sigma) = \sigma \cup \{(a_1, b_1)\}$, which is again a contradiction. Therefore, $(a_i, b_1) \in \sigma$. Further, if $i = 1$, then $\sigma = \mu(\sigma \setminus \{(a_1, b_1)\})$. Therefore, $\sigma = \{(a_i, b_{j_1}), \ldots, (a_i, b_{j_l})\}$ for some $i \in \{2, \ldots, m\}$, $l \geq 2$ and $(a_i, b_1) \in \sigma$.

To prove Claim 3.5, it now suffices to show that $|\sigma| = 2$. Suppose $|\sigma| \geq 3$. Since $|\sigma \setminus \{(a_i, b_1)\}| \geq 2$ and $i \geq 2$, by definition of μ, we have $\mu(\sigma \setminus \{(a_i, b_1)\}) = \sigma$, which is a contradiction to the fact that σ is critical and therefore the result follows. $\square$

From Claim 3.5, all the critical cells for matching μ are of the same dimension, i.e., one dimensional. Moreover, the cardinality of the set C is $(m-1)(n-1)$. Therefore the result follows from Remark 2.11. $\square$

Remark 3.6: Observe that the graph $\underbrace{K_2 \times \ldots \times K_2}_{(r-1)\text{-copies}} \times K_n$ is isomorphic to 2^{r-2} disjoint copies of $K_2 \times K_n$. Therefore, using Lemma 2.7, we get

$$\mathrm{Ind}(\underbrace{K_2 \times \ldots \times K_2}_{(r-1)\text{-copies}} \times K_n) \simeq \bigvee_{(n-1)^{2^{r-2}}} \mathbb{S}^{2^{r-1}-1}.$$

It is thus natural to ask if one can generalise Proposition 3.4 to r-fold product of complete graphs for $r \geq 3$, i.e., if the independence complexes of r-fold product of complete graphs are homotopy equivalent to the wedge sum of spheres. We strongly believe that the independence complexes of r-fold product of complete graphs are homotopy equivalent to wedge of spheres of dimension $2^{r-1} - 1$.

In support of our intuition, we present our computer based computations for the Betti numbers, denoted β_i, of the independence complexes of $K_2 \times K_3 \times K_n$ in Table 1.

n	β_3	β_i, $i \neq 3$
2	4	0
3	14	0
4	30	0
5	52	0
6	80	0

TABLE 1. Betti numbers of $\mathrm{Ind}(K_2 \times K_3 \times K_n)$

These calculations lead us to propose the following conjecture.

Conjecture 3.7. *For $n \geq 2$,*

$$\mathrm{Ind}(K_2 \times K_3 \times K_n) \simeq \bigvee_{(n-1)(3n-2)} \mathbb{S}^3.$$

4. Independence complex of $M_r(K_n)$

This section is devoted to the computation of independence complexes of Mycielskian of graphs. To start with, we compute $\mathrm{Ind}(M_2(G))$ for any graph G. We then focus on the generalised Mycielskian of graphs and determine the homotopy type of $\mathrm{Ind}(M_r(K_n))$ for any n and $r \geq 2$.

Theorem 4.1. *For any graph G, $\mathrm{Ind}(M_2(G)) \simeq \Sigma(\mathrm{Ind}(G))$.*

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ and let $w = (v_1, 2) = \cdots = (v_n, 2)$. Let $K = st_{\mathrm{Ind}(M_2(G))}(w) \cap SC_{\mathrm{Ind}(M_2(G))}(\{(v_1, 1), \dots, (v_n, 1)\})$. Since $N(w) = \{(v_1, 1), \dots, (v_n, 1)\} \in \mathrm{Ind}(M_2(G))$, Lemma 2.6 implies that

$$\mathrm{Ind}(M_2(G)) \simeq \Sigma(K).$$

Let H be the subgraph of $M_2(G)$ induced by $\{(v_1, 0), \dots, (v_n, 0)\}$. Clearly, $H \cong G$ and therefore $\mathrm{Ind}(H) \cong \mathrm{Ind}(G)$. We now show that $K = \mathrm{Ind}(H)$.

Let $\sigma \in \mathrm{Ind}(H)$ and $(v_i, 0) \in \sigma$, then $N((v_i, 0)) \cap \sigma = \emptyset$. Since $N((v_i, 1)) \subseteq N((v_i, 0)) \cup \{w\}$, we have $N((v_i, 1)) \cap \sigma = \emptyset$ thereby implying that $\sigma \in st_{\mathrm{Ind}(M_2(G))}(\{(v_i, 1)\})$. Since $\sigma \subseteq V(H)$, we see that $\sigma \cup \{w\} \in \mathrm{Ind}(M_2(G))$. Therefore, $\sigma \in st_{\mathrm{Ind}(M_2(G))}(\{w\}) \cap SC_{\mathrm{Ind}(M_2(G))}(\{(v_1, 1), \dots, (v_n, 1)\})$ and hence $\mathrm{Ind}(H) \subseteq K$.

Now suppose that $\sigma \in K$. For each i, w is adjacent to $(v_i, 1)$ in $M_2(G)$, therefore $\sigma \cap \{w, (v_i, 1)\} = \emptyset$ for all i, and hence $K \subseteq \mathrm{Ind}(H)$. $\square$

We now list a few results which will be used in this section for the computation of the independence complex of the generalised Mycielskian of complete graphs.

Lemma 4.2 ([7, Lemma 3.4])**.** *Let G be a graph and $u, u' \in V(G)$ such that $N(u) \subseteq N(u')$. Then,*

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus u').$$

Lemma 4.3 ([17, Proposition 2.10]). *Let G be a graph and let $\{a,b\} \in \mathrm{Ind}(G)$. If $\mathrm{Ind}(G - N[\{a,b\}])$ is collapsible, then $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(\tilde{G})$, where $V(\tilde{G}) = V(G)$ and $E(\tilde{G}) = E(G) \cup \{(a,b)\}$. In particular, $\mathrm{Ind}(G) \simeq \mathrm{Ind}(\tilde{G})$.*

Lemma 4.4 ([12, Lemma 2.1]). *Let G be graph and v be a simplicial vertex*[1] *of G. Let $N(v) = \{w_1, w_2, \ldots, w_k\}$. Then*

$$\mathrm{Ind}(G) \simeq \bigvee_{i=1}^{k} \Sigma(\mathrm{Ind}(G - N[w_i])).$$

Definition 4.5. *Let $p : X \to Y$ and $q : X \to Z$ be two continuous maps. The pushout of the diagram $Y \xleftarrow{p} X \xrightarrow{q} Z$ is the space*

$$\left(Y \bigsqcup Z\right)/\sim,$$

where $\sim$ denotes the equivalence relation $p(x) \sim q(x)$ for $x \in X$.

The homotopy pushout of $Y \xleftarrow{p} X \xrightarrow{q} Z$ is the space $\big(Y \sqcup (X \times I) \sqcup Z\big)/\sim$, where $\sim$ denotes the equivalence relation $(x,0) \sim p(x)$, and $(x,1) \sim q(x)$ for $x \in X$. It can be shown that homotopy pushouts of any two homotopy equivalent diagrams are homotopy equivalent.

Remark 4.6: If spaces are CW complexes and maps are subcomplex inclusions, then their homotopy pushout and pushout spaces are equivalent up to homotopy. For an elaborate discussion of these results, we refer interested readers to [5, Chapter 7].

Lemma 4.7. *Let X be a simplicial complex and $v \in V(X)$. Let $Y = \{\sigma \in X \mid v \notin \sigma\}$ be a subcomplex of X. If $lk_X(v)$ is contractible, then $X \simeq Y$.*

Proof. Let $A = lk_X(v)$ and let Z be the homotopy pushout of the diagram $A \xleftarrow{=} A \hookrightarrow Y$. Since $A \times I$ is homotopy equivalent to A, $Y \simeq Z$. Also, contractibility of A implies that Z is of the same homotopy type as $Z/(A \times \{1\})$. Therefore, $Y \simeq Z \simeq Z/(A \times \{1\})$ which is homeomorphic to X. $\square$

Lemma 4.8. *Let $n \geq 2$ and $X_1, X_2, \ldots, X_n$ be simplicial complexes. If each X_i is contractible and for each $j \in \{2, 3, \ldots, n\}$, $\big(\bigcup_{i=1}^{j-1} X_i\big) \cap X_j \simeq \bigvee_r \mathbb{S}^k$, then $X_1 \cup X_2 \cup \cdots \cup X_n \simeq \bigvee_{(n-1)r} \mathbb{S}^{k+1}$.*

Proof. Observe that $X_1 \cup X_2$ is the pushout of the diagram $X_1 \hookleftarrow X_1 \cap X_2 \hookrightarrow X_2$, where $\hookrightarrow$ denotes inclusion maps. From Remark 4.6, the homotopy pushout and pushout of $X_1 \hookleftarrow X_1 \cap X_2 \hookrightarrow X_2$ are homotopy equivalent to each other. Further, the homotopy pushout of $X_1 \hookleftarrow X_1 \cap X_2 \hookrightarrow X_2$ is homotopy equivalent to the homotopy pushout of $\{\text{point}\} \longleftarrow \bigvee_r \mathbb{S}^k \longrightarrow \{\text{point}\}$ (since X_1 and X_2 are contractible and $X_1 \cap X_2 \simeq \bigvee_r \mathbb{S}^k$). Moreover, homotopy pushout of $\{\text{point}\} \longleftarrow \bigvee_r \mathbb{S}^k \longrightarrow \{\text{point}\}$ is homotopy equivalent to $\Sigma\big(\bigvee_r \mathbb{S}^k\big)$. Therefore, $X_1 \cup X_2 \simeq \bigvee_r \mathbb{S}^{k+1}$.

[1]A vertex v of G is called *simplicial* if the subgraph induced by $N(v)$ is a complete graph.

Let $n \geq 3$. Inductively assume that for any $2 \leq t < n$, $\bigcup_{i=1}^{t} X_i \simeq \bigvee_{(t-1)r} \mathbb{S}^{k+1}$. In particular, $\bigcup_{i=1}^{n-1} X_i \simeq \bigvee_{(n-2)r} \mathbb{S}^{k+1}$. Further, the pushout of the diagram $\bigcup_{i=1}^{n-1} X_i \hookleftarrow \big(\bigcup_{i=1}^{n-1} X_i\big) \cap X_n \hookrightarrow X_n$ is the space $\bigcup_{i=1}^{n} X_i$. Thus, from Remark 4.6, $\bigcup_{i=1}^{n} X_i$ is homotopy equivalent to the homotopy pushout of the diagram $\bigvee_{(n-2)r} \mathbb{S}^{k+1} \longleftarrow \bigvee_{r} \mathbb{S}^{k} \longrightarrow \{\text{point}\}$ which is homotopy equivalent to $\bigvee_{(n-2)r+r} \mathbb{S}^{k+1}$. □

Lemma 4.9. *Let $n \geq 2$ and $X_1, X_2, \ldots, X_n$ be simplicial complexes. If for each $i \in \{1, 2, \ldots, n\}$, $X_i \simeq \bigvee_r \mathbb{S}^k$ and for each $j \in \{2, 3, \ldots, n\}$, $\big(\bigcup_{i=1}^{j-1} X_i\big) \cap X_j$ is contractible, then $X_1 \cup X_2 \cup \cdots \cup X_n \simeq \bigvee_{nr} \mathbb{S}^k$.*

Proof. Using similar arguments as in the proof of Lemma 4.8, we get that $X_1 \cup X_2$ is homotopy equivalent to the homotopy pushout of $\bigvee_r \mathbb{S}^k \hookleftarrow \{\text{point}\} \hookrightarrow \bigvee_r \mathbb{S}^k$ (since $X_1 \simeq \bigvee_r \mathbb{S}^k \simeq X_2$ and $X_1 \cap X_2$ is contractible).

Further, homotopy pushout of $\bigvee_r \mathbb{S}^k \hookleftarrow \{\text{point}\} \hookrightarrow \bigvee_r \mathbb{S}^k$ is homotopy equivalent to $\bigvee_{r+r} \mathbb{S}^k$. Thus, $X_1 \cup X_2 \simeq \bigvee_{2r} \mathbb{S}^k$. As before, the result now follows from induction. □

Proposition 4.10. *Let $r \geq 0$ and $n \geq 2$. Then*

$$\operatorname{Ind}(K_n \times L_r) \simeq \begin{cases} \bigvee\limits_{(n-1)^{k+1}} \mathbb{S}^{2k} & \text{if } r = 3k, \\ \{\text{point}\} & \text{if } r = 3k+1, \\ \bigvee\limits_{(n-1)^{k+1}} \mathbb{S}^{2k+1} & \text{if } r = 3k+2. \end{cases}$$

Proof. Let $r = 3k + t$ for some $t \in \{0, 1, 2\}$ and $k \geq 0$. We prove this result by induction on k.

To prove the base step, let $k = 0$. We show that the result holds for $t \in \{0, 1, 2\}$. If $t = 0$, then $K_n \times L_0$ isomorphic to K_n implies

$$\operatorname{Ind}(K_n \times L_0) \cong \operatorname{Ind}(K_n) = \bigvee_{n-1} \mathbb{S}^0.$$

For $t = 1$, let H_1 be the induced subgraph of $K_n \times L_1$ with vertex set $\{(i, 1) \mid 1 \leq i \leq n\}$. Since H_1 does not have any edge, $\operatorname{Ind}(H_1) \simeq \{\text{point}\}$. Observe that, in $K_n \times L_1$, $N((i, 1)) \subseteq N((i, 0))$ for each $i \in \{1, 2, \ldots, n\}$. Now repeated use of Lemma 4.2 for each i gives us $\operatorname{Ind}(K_n \times L_1) \simeq \operatorname{Ind}(H_1) \simeq \{\text{point}\}$. This proves the result for $k = 0$ and $t = 1$.

Finally, if $t = 2$, let H_2 be the induced subgraph of $K_n \times L_2$ with vertex set $\{(i, j) \mid 1 \leq i \leq n,\ 1 \leq j \leq 2)\}$. Clearly, $H_2 \cong K_2 \times K_n$. We observe that in $K_n \times L_2$, $N((i, 2)) \subseteq N((i, 0))$ for each $i \in \{1, 2, \ldots, n\}$. By Lemma 4.2 and Proposition 3.4, we conclude that

$$\operatorname{Ind}(K_n \times L_2) \simeq \operatorname{Ind}(K_2 \times K_n) \simeq \bigvee_{n-1} \mathbb{S}^1.$$

Inductively assume that the result is true for $k < s$ and $t \in \{0, 1, 2\}$. We now show that the result holds for $k = s > 0$ and every $t \in \{0, 1, 2\}$.

Let H be the induced subgraph of $K_n \times L_{3s+t}$ with vertex set $V(K_n \times L_{3s+t}) \setminus \{(i, 3s+t-2) \mid 1 \leq i \leq n\}$. Further, in $K_n \times L_{3s+t}$, $N((i, 3s+t)) \subseteq N((i, 3s+t-2))$. Thus, using Lemma 4.2, we have that $\mathrm{Ind}(K_n \times L_{3s+t}) \simeq \mathrm{Ind}(H)$. Now observe that, $H \cong (K_2 \times K_n) \bigsqcup (K_n \times L_{3s+t-3})$. Using Lemma 2.7 and Proposition 3.4, we get

$$\begin{aligned}
\mathrm{Ind}(K_n \times L_{3s+t}) &\simeq \mathrm{Ind}(H) \\
&\simeq \mathrm{Ind}(K_2 \times K_n) * \mathrm{Ind}(K_n \times L_{3s+t-3}) \\
&\simeq \Big(\bigvee_{n-1} \mathbb{S}^1\Big) * \big(\mathrm{Ind}(K_n \times L_{3(s-1)+t})\big).
\end{aligned}$$

By induction hypothesis, we get

$$\mathrm{Ind}(K_n \times L_{3s+t}) \simeq \begin{cases} \big(\bigvee\limits_{n-1} \mathbb{S}^1\big) * \big(\bigvee\limits_{(n-1)^s} \mathbb{S}^{2(s-1)}\big) & \text{if } t = 0, \\ \big(\bigvee\limits_{n-1} \mathbb{S}^1\big) * \{\text{point}\} & \text{if } t = 1, \\ \big(\bigvee\limits_{n-1} \mathbb{S}^1\big) * \big(\bigvee\limits_{(n-1)^s} \mathbb{S}^{2(s-1)+1}\big) & \text{if } t = 2. \end{cases}$$

$$\simeq \begin{cases} \bigvee\limits_{(n-1)^{s+1}} \mathbb{S}^{2s} & \text{if } t = 0, \\ \{\text{point}\} & \text{if } t = 1, \\ \bigvee\limits_{(n-1)^{s+1}} \mathbb{S}^{2s+1} & \text{if } t = 2. \end{cases}$$

This completes the proof of Proposition 4.10. □

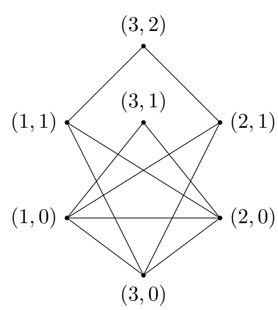


FIGURE 3. $I^3_{3,2}$

We fix a natural number $n \geq 3$, and define a few notations that would be used in the rest of this section.

(1) Let $V(K_n) = \{1, 2, \ldots, n\}$.
(2) In $M_r(K_n)$, the equivalence class of vertices with second coordinate as r is denoted by w_r.
(3) For $r \geq 3$, let $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r-1\}$. We define $I^n_{i,j}$ to be the subgraph of $M_r(K_n)$ induced by $(V(M_j(K_n)) \setminus w_j) \cup \{(i, j)\}$. Also, let $I^n_{i,0}$ be the subgraph of $M_r(K_n)$ induced by the vertex $(i, 0)$.

Example 4.11. *$I^3_{3,2}$ is the induced subgraph of $M_r(K_3)$ on the vertex set $\{(s,t)\ :\ 1 \leq s \leq 3,\ 0 \leq t \leq 1\} \cup \{(3,2)\}$ (see Figure 3).*

Since $n \geq 3$ is fixed for the rest of this section, we would write $I_{i,j}$ to denote $I^n_{i,j}$ for the simplicity of notation.

Lemma 4.12. *Let $I_{1,t}$ be as defined above, then*

$$\mathrm{Ind}(I_{1,t}) \simeq \begin{cases} \bigvee\limits_{n-1} \mathbb{S}^0 & \textit{if } t = 1, \\ \bigvee\limits_{n-1} \mathbb{S}^1 & \textit{if } t = 2, \\ \{\text{point}\} & \textit{if } t = 3. \end{cases}$$

Proof. In $I_{1,1}$, $N((1,0)) = \{(2,0), \ldots, (n,0)\} = N((1,1))$ and therefore by Lemma 4.2, $\mathrm{Ind}(I_{1,1}) \simeq \mathrm{Ind}(I_{1,1} \setminus \{(1,1)\}) \cong \mathrm{Ind}(K_n) \simeq \bigvee_{n-1} \mathbb{S}^0$.

Recall that a vertex v is simplicial if the subgraph induced by $N(v)$ is a complete graph. In $I_{1,2}$, $N((1,1)) = \{(2,0), \ldots, (n,0)\}$ and therefore $(1,1)$ is a simplicial vertex. Moreover, for each $2 \leq i \leq n$, $V(I_{1,2}) \setminus N[(i,0)] = \{(1,2), (i,1)\}$, implying that $I_{1,2} - N[(i,0)] \cong K_2$. Therefore, using Lemma 4.4, we get that

$$\mathrm{Ind}(I_{1,2}) \simeq \bigvee_{n-1} \Sigma(\mathrm{Ind}(K_2)) \simeq \bigvee_{n-1} \Sigma(\mathbb{S}^0) \simeq \bigvee_{n-1} \mathbb{S}^1.$$

Since the graph $I_{1,3} - N[\{(1,1),(2,1)\}]$ contains an isolated vertex $(1,3)$, $\mathrm{Ind}(I_{1,3} - N[\{(1,1),(2,1)\}])$ is a cone by Lemma 2.7, and hence collapsible. Using Lemma 4.3, $\mathrm{Ind}(I_{1,3}) \simeq \mathrm{Ind}(I'_{1,3})$, where $I'_{1,3}$ is the graph with $V(I'_{1,3}) = V(I_{1,3})$ and $E(I'_{1,3}) = E(I_{1,3}) \cup \{((1,1),(2,1))\}$. We repeat this process for all pair of vertices $((i,1),(j,1))$ $1 \leq i \neq j \leq n$ and apply Lemma 4.3, which thereby implies that $\mathrm{Ind}(I_{1,3}) \simeq \mathrm{Ind}(\tilde{I}_{1,3})$, where $V(\tilde{I}_{1,3}) = V(I_{1,3})$ and $E(\tilde{I}_{1,3}) = E(I_{1,3}) \cup \{((i,1),(j,1))\ :\ 1 \leq i \neq j \leq n\}$. For each $1 \leq i \leq n$, $\tilde{I}_{1,3} - N[\{(i,1)\}]$ contains an isolated vertex $(i,0)$ and therefore $\mathrm{Ind}(\tilde{I}_{1,3} - N[\{(i,1)\}])$ is collapsible by Lemma 2.7. Now, using the fact that $(1,2)$ is a simplicial vertex in $\tilde{I}_{1,3}$ and $\mathrm{Ind}(\tilde{I}_{1,3} - N[\{(i,1)\}]) \cong \mathrm{Ind}(\tilde{I}_{1,3} - N[\{(j,1)\}])$ for all $2 \leq i \neq j \leq n$, by Lemma 4.4, $\mathrm{Ind}(\tilde{I}_{1,3})$ is contractible. Hence, $\mathrm{Ind}(I_{1,3})$ is contractible. $\square$

We now generalise Lemma 4.12 and compute the homotopy type of independence complex of $I_{1,t}$, for any natural number t.

Lemma 4.13. *Let $t \geq 6$. Then*

$$\mathrm{Ind}(I_{1,t-2}) \simeq \begin{cases} \bigvee\limits_{(n-1)^k} \mathbb{S}^{2(k-1)+1} & \textit{if } t = 3k+1, \\ \{\text{point}\} & \textit{if } t = 3k+2, \\ \bigvee\limits_{(n-1)^k} \mathbb{S}^{2(k-1)} & \textit{if } t = 3k. \end{cases}$$

Proof. To prove this, we first construct a graph $\tilde{I}_{1,t-2}$ that contains $I_{1,t-2}$ as a subgraph such that $(1, t-3)$ is a simplicial vertex in $\tilde{I}_{1,t-2}$ and $\text{Ind}(I_{1,t-2}) \simeq \text{Ind}(\tilde{I}_{1,t-2})$. Observe that the vertex $(1, t-2)$ is an isolated vertex in $I_{1,t-2} - N[\{(1, t-4), (2, t-4)\}]$ and hence $\text{Ind}(I_{1,t-2} - N[\{(1, t-4), (2, t-4)\}])$ is collapsible. By Lemma 4.3, $\text{Ind}(I_{1,t-2}) \simeq \text{Ind}(H)$, where $V(H) = V(I_{1,t-2})$ and $E(H) = E(I_{1,t-2}) \cup \{((1, t-4), (2, t-4))\}$. By repeating this process for all pairs (i, j), $1 \leq i \neq j \leq n$, we get the graph $\tilde{I}_{1,t-2}$ such that $\text{Ind}(I_{1,t-2}) \simeq \text{Ind}(\tilde{I}_{1,t-2})$, where $V(\tilde{I}_{1,t-2}) = V(I_{1,t-2})$ and $E(\tilde{I}_{1,t-2}) = E(I_{1,t-2}) \cup \{((i, t-4), (j, t-4)) : 1 \leq i \neq j \leq n\}$. Since $N((1, t-3)) = \{(2, t-4), \ldots, (n, t-4)\}$, the vertex $(1, t-3)$ is a simplicial vertex in $\tilde{I}_{1,t-2}$. From Lemma 4.4, we have

$$\text{Ind}(I_{1,t-2}) \simeq \text{Ind}(\tilde{I}_{1,t-2}) \simeq \bigvee_{i=2}^{n} \Sigma\big(\text{Ind}(\tilde{I}_{1,t-2} - N[\{(i, t-4)\}])\big).$$

Observe that for each $2 \leq i \leq n$, the graph $\tilde{I}_{1,t-2} - N[\{(i, t-4)\}]$ is isomorphic to $I_{i,t-5} \sqcup K_2$, where K_2 appears because of the edge $((1, t-2), (i, t-3))$ in $\tilde{I}_{1,t-2} - N[\{(i, t-4)\}]$. We note that for any j, $I_{i,j} \cong I_{l,j}$ and therefore

$$\text{Ind}(I_{1,t-2}) \simeq \bigvee_{(n-1)} \Sigma\Big(\text{Ind}(I_{1,t-5} \sqcup K_2)\Big) \simeq \bigvee_{(n-1)} \Sigma^2\Big(\text{Ind}(I_{1,t-5})\Big). \tag{4.1}$$

We consider the following three cases.

Case 1: $t = 3k$.

In this case, since $t - 2 = 3(k-1) + 1$, using equation (4.1) and Lemma 4.12, we conclude that

$$\begin{aligned}\text{Ind}(I_{1,t-2}) &\simeq \bigvee_{(n-1)^{k-1}} \Sigma^{2(k-1)}(\text{Ind}(I_{1,1})) \\ &\simeq \bigvee_{(n-1)^{k-1}} \Sigma^{2(k-1)}(\bigvee_{n-1} \mathbb{S}^0) \simeq \bigvee_{(n-1)^k} \mathbb{S}^{2(k-1)}.\end{aligned}$$

Case 2: $t = 3k + 1$.

In this case, $t - 2 = 3(k-1) + 2$. Again by Lemma 4.12 and equation (4.1), we have

$$\begin{aligned}\text{Ind}(I_{1,t-2}) &\simeq \bigvee_{(n-1)^{k-1}} \Sigma^{2(k-1)}(\text{Ind}(I_{1,2})) \\ &\simeq \bigvee_{(n-1)^{k-1}} \Sigma^{2(k-1)}(\bigvee_{n-1} \mathbb{S}^1) \simeq \bigvee_{(n-1)^k} \mathbb{S}^{2(k-1)+1}.\end{aligned}$$

Case 3: $t = 3k + 2$.

In this case, since $t - 2 = 3(k-1) + 3$,

$$\text{Ind}(I_{1,t-2}) \simeq \bigvee_{(n-1)^{k-1}} \Sigma^{2(k-1)}(\text{Ind}(I_{1,3})).$$

By Lemma 4.12, $\text{Ind}(I_{1,3})$ is contractible and so is $\text{Ind}(I_{1,t-2})$. $\square$

We are now ready to prove the main result of this section. Firstly, note that $M_r(K_2)$ is isomorphic to the odd cycle C_{2r+1}, for which the independence complex has been computed by Kozlov in [13]. Here, we determine the homotopy type of $\text{Ind}(M_r(K_n))$ for $n > 2$ and any r.

Theorem 4.14. *Let $r \geq 2$ be a positive integer. Then,*

$$\text{Ind}(M_r(K_n)) \simeq \begin{cases} \bigvee_{(n-1)^k} \mathbb{S}^{2k-1} & \text{if } r = 3k, \\ \bigvee_{n(n-1)^k} \mathbb{S}^{2k} & \text{if } r = 3k+1, \\ \bigvee_{(n-1)^{(k+1)}} \mathbb{S}^{2k+1} & \text{if } r = 3k+2. \end{cases}$$

Proof. If $r = 2$, then Theorem 4.1 implies that $\text{Ind}(M_r(K_n)) \simeq \Sigma(\text{Ind}(K_n))$, and hence the result follows. So we assume that $r \geq 3$. In $M_r(K_n)$, $N(w_r) = \{(1, r-1), \ldots, (n, r-1)\}$ and there is no edge among the vertices of $N(w_r)$. Therefore, from Lemma 2.6

$$\begin{aligned} \text{Ind}(M_r(K_n)) \simeq \Sigma\big(&st_{\text{Ind}(M_r(K_n))}(w_r) \cap \\ &SC_{\text{Ind}(M_r(K_n))}(\{(1, r-1), \ldots, (n, r-1)\})\big). \end{aligned} \tag{4.2}$$

Let K denote the complex

$$st_{\text{Ind}(M_r(K_n))}(w_r) \cap SC_{\text{Ind}(M_r(K_n))}(\{(1, r-1), \ldots, (n, r-1)\}).$$

Claim 4.15. $K = \bigcup_{i=1}^{n} \text{Ind}(I_{i,r-2})$.

Let $\sigma \in K$. If $\sigma \cap \{(1, r-2), \ldots, (n, r-2)\} = \emptyset$, then $\sigma \in \bigcup_{i=1}^{n} \text{Ind}(I_{i,r-2})$. On the other hand if $\sigma \cap \{(1, r-2), \ldots, (n, r-2)\} \neq \emptyset$, then there exists a unique i such that $(i, r-2) \in \sigma$ and in this case $\sigma \in \text{Ind}(I_{i,r-2})$. Hence, $K \subseteq \bigcup_{i=1}^{n} \text{Ind}(I_{i,r-2})$. If $\tau \in \bigcup_{i=1}^{n} \text{Ind}(I_{i,r-2})$, then $\tau \in \text{Ind}(I_{i,r-2})$ for some i, therefore $\tau \in st_{\text{Ind}(M_r(K_n))}((i, r-1)) \cap st_{\text{Ind}(M_r(K_n))}(w_r)$. This completes the proof of Claim 4.15.

For $1 \leq i \neq l \leq n$, observe that $I_{i,j} \cap I_{l,j} \cong K_n \times L_{j-1}$, therefore $\text{Ind}(I_{i,j} \cap I_{l,j}) \simeq \text{Ind}(K_n \times L_{j-1})$. Also, for any arbitrary graph G and $A, B \subseteq V(G)$, $\text{Ind}(G[A \cap B]) = \text{Ind}(G[A]) \cap \text{Ind}(G[B])$, and hence $\text{Ind}(I_{i,j} \cap I_{l,j}) = \text{Ind}(I_{i,j}) \cap \text{Ind}(I_{l,j})$. Further, $\big(\bigcup_{s=1}^{i} \text{Ind}(I_{s,j})\big) \bigcap \text{Ind}(I_{i+1,j}) = \text{Ind}(I_{i,j}) \cap \text{Ind}(I_{i+1,j}) \simeq \text{Ind}(K_n \times L_{j-1})$ for all $1 \leq i \leq n-1$.

Case 1: $r = 3k+1$.

From Lemmas 4.12 and 4.13, $\text{Ind}(I_{i,r-2}) \simeq \bigvee_{(n-1)^k} \mathbb{S}^{2(k-1)+1}$. Proposition 4.10 implies that $\text{Ind}(K_n \times L_{r-3})$ is contractible and therefore by using Claim 4.15 and Lemma 4.9, we conclude that $K \simeq \bigvee_{n(n-1)^k} \mathbb{S}^{2(k-1)+1}$. Thus, from equation (4.2),

$$\text{Ind}(M_r(K_n)) \simeq \Sigma(K) \simeq \bigvee_{n(n-1)^k} \mathbb{S}^{2k}.$$

Case 2 : $r = 3k+2$.

In this case, $\text{Ind}(I_{i,r-2})$ is contractible from Lemmas 4.12 and 4.13. Since, $r-3 = 3(k-1)+2$, $\text{Ind}(K_n \times L_{r-3}) \simeq \bigvee_{(n-1)^k} \mathbb{S}^{2k-1}$ from Lemma 4.10. Hence by using Lemma 4.8 and Claim 4.15, we conclude that $K \simeq \bigvee_{(n-1)(n-1)^k} \mathbb{S}^{2k}$. Thus,

$$\text{Ind}(M_r(K_n)) \simeq \bigvee_{(n-1)^{k+1}} \mathbb{S}^{2k+1}.$$

Case 3. $r = 3k$.

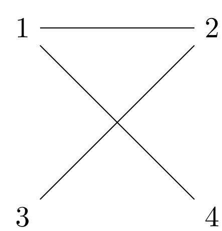
First we show that $K \simeq \text{Ind}(K_n \times L_{r-3})$. Observe that $lk_K((1, r-2)) = \text{Ind}(I_{1,r-3})$. Using Lemmas 4.12 and 4.13, we get that $\text{Ind}(I_{1,r-3})$ is contractible. Hence Lemma 4.7 implies that $K \simeq K^1 := K \setminus \{\sigma \in K : (1, r-3) \in \sigma\}$. Now $lk_{K^1}((2, r-2)) = \text{Ind}(I_{2,r-3}) \cong \text{Ind}(I_{1,r-3})$ and therefore $K^1 \simeq K^2 := K^1 \setminus \{\sigma \in K^1 : (2, r-3) \in \sigma\}$. Repeating this process for all $(i, r-2), 3 \leq i \leq n$, we conclude that $K \simeq K^n := \{\sigma \in K : (1, r-2), \ldots, (n, r-2) \notin \sigma\}$. It is easy to check that $K^n \cong \text{Ind}(K_n \times L_{r-3})$. From Proposition 4.10, we conclude that

$$\text{Ind}(M_r(K_n)) \simeq \Sigma(\text{Ind}(K_n \times L_{r-3})) \simeq \Sigma(\bigvee_{(n-1)^k} \mathbb{S}^{2(k-1)}) \simeq \bigvee_{(n-1)^k} \mathbb{S}^{2k-1},$$

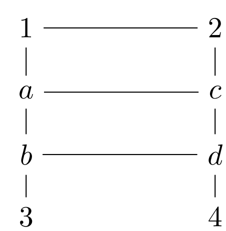
and this completes the proof. □

5. Independence Complex of Subdivision of Cycles

Let G be a graph that contains a crossing of two edges as in Figure 4a. In this section we prove that if we replace a crossing in graph G with the structure as in Figure 4b, then the independence complex of H is the suspension of the independence complex of G.



(A) A crossing in G



(B) 2 ladder in H

FIGURE 4. Replacing a crossing in G by a 2 ladder to obtain H

Theorem 5.1. *Let G be a graph that contains graph depicted in Figure 4a as a subgraph. If H is the graph with $V(H) = V(G) \sqcup \{a, b, c, d\}$, $E(H) = (E(G) \setminus \{(1,4), (2,3)\}) \cup \{(1,a), (a,b), (b,3), (2,c), (c,d), (d,4), (a,c), (b,d)\}$ obtained from G as shown in Figure 4b, then*

$$\text{Ind(H)} \simeq \Sigma(\text{Ind(G)}).$$

Proof. Observe that $\{a, d\}$ and $\{b, c\}$ are simplices of $\mathrm{Ind}(H)$. Let $K = \mathrm{Ind}(H)$, $K_1 = SC_K(\{a, d\})$ and $K_2 = SC_K(\{b, c\})$. From Lemma 2.4, K_1 and K_2 are contractible subcomplexes of K.

We first show that $K = K_1 \cup K_2$. Clearly, $K_1 \cup K_2$ is a subcomplex of K. Let $\sigma \in K$. If $\{a, b, c, d\} \cap \sigma \neq \emptyset$, then by definition $\sigma \in K_1$ or K_2. Therefore we assume that $\{a, b, c, d\} \cap \sigma = \emptyset$. If $1 \notin \sigma$, then $\sigma \in st_K(\{a\})$. If $1 \in \sigma$, then $2 \notin \sigma$ and thereby implying that $\sigma \in st_K(\{c\})$.

Claim 5.2. $K_1 \cap K_2 = \mathrm{Ind}(G)$.

Let $\sigma \in K_1 \cap K_2$. Clearly, $\{a, b, c, d\} \cap \sigma = \emptyset$. To show that $\sigma \in \mathrm{Ind}(G)$, it is enough to show that $\{1, 4\} \nsubseteq \sigma$ and $\{2, 3\} \nsubseteq \sigma$. However, $\{1, 4\} \notin K_1$ implies that $\{1, 4\} \nsubseteq \sigma$. Similarly, $\{2, 3\} \notin K_2$ implies that $\{2, 3\} \nsubseteq \sigma$.

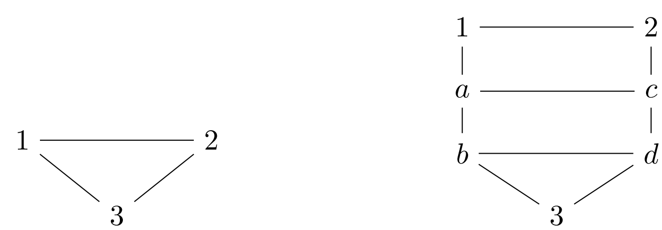
Now, let $\sigma \in \mathrm{Ind}(G)$. Since $\{1, 4\} \nsubseteq \sigma$, σ is in $st_K(\{a\})$ or $st_K(\{d\})$. Moreover, $\{2, 3\} \nsubseteq \sigma$ implies that either σ is in $st_K(\{b\})$ or is in $st_K(\{c\})$. Therefore, $\sigma \in K_1 \cap K_2$.

Thus result follows from Lemma 2.5. $\square$

We note that the proof of Theorem 5.1 also holds if we assume that vertices 3 and 4 are the same (cf. Figure 5), i.e., $3 = 4$. We, therefore, have the following result as a special case.

Theorem 5.3 ([17, Section 3.3.1])**.** *Let G be a graph that contains triangle depicted in Figure 5a as a subgraph. If H is the graph with $V(H) = V(G) \sqcup \{a, b, c, d\}$, $E(H) = (E(G) \setminus \{(1, 3), (2, 3)\}) \cup \{(1, a), (a, b), (b, 3), (2, c), (c, d), (d, 3), (a, c), (b, d)\}$ obtained from G as shown in Figure 5b, then*

$$\mathrm{Ind(H)} \simeq \Sigma(\mathrm{Ind(G)}).$$



(A) A triangle in G (B) 2 ladder in H

FIGURE 5. Replacing an edge in G by a 2 ladder to obtain H

Before proceeding further, we would like to point out that Skwarski has considered a similar construction as in Figure 5 in [17, Section 3.3.1].

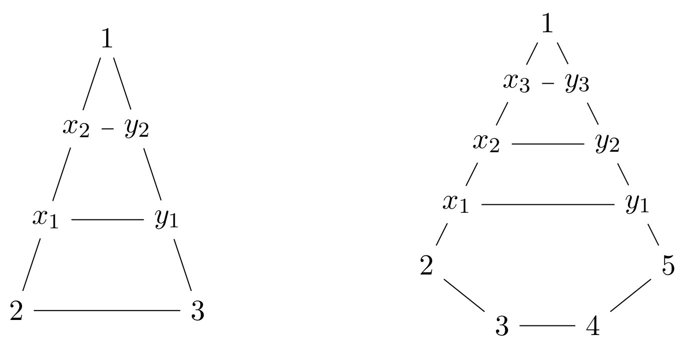
We now record a straightforward observation that follows from Theorem 5.3.

Corollary 5.4. *Let G be a graph with Figure 5b as an induced subgraph. Then the independence complex of G has the homotopy type of a suspension.*

As an application of Theorem 5.1 and Theorem 5.3, we compute the homotopy type of the independence complexes of a particular family of graphs.

Let $C_n^0 \equiv C_n$ be the cycle graph on the vertex set $\{1, 2, \dots, n\}$. Let C_n^1 be the graph obtained from C_n by subdividing the edges adjacent to 1 and adding an edge between the newly created vertices. Let x_1, y_1 be the vertices of $V(C_n^1) \setminus V(C_n^0)$. We iteratively define the graph C_n^j to be the graph obtained from C_n^{j-1} as per the above construction. We note that $V(C_n^j) = \{1, 2, \dots, n\} \cup \{x_1, x_2, \dots, x_j\} \cup \{y_1, y_2, \dots, y_j\}$, i.e., $|V(C_n^j)| = n + 2j$ and $E(C_n^j) = (E(C_n^0) \setminus \{(1, 2), (1, n)\}) \cup \{(x_i, y_i) : 1 \le i \le j\} \cup \{(x_i, x_{i+1}), (y_i, y_{i+1}) : 1 \le i \le j-1\} \cup \{(1, x_j), (1, y_j), (2, x_1), (n, y_1)\}$.

For example, Figure 6 shows C_3^2 and C_5^3.



(A) C_3^2 (B) C_5^3

FIGURE 6. Subdivision of cycles

Corollary 5.5. *Let $i, r \ge 1$, then*

$$\mathrm{Ind}(C_{3r}^i) \simeq \begin{cases} \mathbb{S}^{r-1+j} \bigvee \mathbb{S}^{r-1+j} & \text{if } i = 2j, \\ \{\text{point}\} & \text{otherwise}, \end{cases}$$

$$\mathrm{Ind}(C_{3r+1}^i) \simeq \begin{cases} \mathbb{S}^{r-1+j} \bigvee \mathbb{S}^{r-1+j} & \text{if } i = 2j-1, \\ \{\text{point}\} & \text{otherwise}, \end{cases}$$

and

$$\mathrm{Ind}(C_{3r+2}^i) \simeq \begin{cases} \mathbb{S}^{r} \bigvee \mathbb{S}^{r} & \text{if } i = 1, \\ \mathbb{S}^{r+j} \bigvee \mathbb{S}^{r+j} & \text{if } i = 2j \text{ or } 2j+1,\ j \ge 1. \end{cases}$$

Proof. Let P_n be the path graph on n vertices with $n-1$ edges. From [14, Proposition 11.16], we know that

$$\mathrm{Ind}(P_n) \simeq \begin{cases} \mathbb{S}^{r-1} & \text{if } n = 3r, \\ \{\text{point}\} & \text{if } n = 3r+1, \\ \mathbb{S}^{r} & \text{if } 3r+2. \end{cases} \tag{5.1}$$

We give the proof by induction on i. Observe that in C_n^i, vertex 1 is a simplicial vertex for each $i \geq 1$. Therefore, using Lemma 4.4 and equation 5.1 we get

$$\begin{aligned}
\mathrm{Ind}(C_n^1) &\simeq \Sigma\big(\mathrm{Ind}(C_n^1 - \{1, 2, x_1, y_1\})\big)\bigvee\Sigma\big(\mathrm{Ind}(C_n^1 - \{1, n, x_1, y_1\})\big)\\
&\simeq \Sigma\big(\mathrm{Ind}(P_{n-2})\big)\bigvee\Sigma\big(\mathrm{Ind}(P_{n-2})\big)\\
&\simeq \begin{cases} \{\text{point}\} & \text{if } n = 3r,\\ \mathbb{S}^r\bigvee\mathbb{S}^r & \text{if } n = 3r+1, 3r+2. \end{cases}
\end{aligned}$$

Similarly, using Lemma 4.4 and equation 5.1 we get

$$\begin{aligned}
\mathrm{Ind}(C_n^2) &\simeq \Sigma\big(\mathrm{Ind}(C_n^2 - \{1, x_1, x_2, y_2\})\big)\bigvee\Sigma\big(\mathrm{Ind}(C_n^1 - \{1, x_2, y_1, y_2\})\big)\\
&\simeq \Sigma\big(\mathrm{Ind}(P_n)\big)\bigvee\Sigma\big(\mathrm{Ind}(P_n)\big)\\
&\simeq \begin{cases} \mathbb{S}^r\bigvee\mathbb{S}^r & \text{if } n = 3r,\\ \{\text{point}\} & \text{if } n = 3r+1,\\ \mathbb{S}^{r+1}\bigvee\mathbb{S}^{r+1} & \text{if } n = 3r+2. \end{cases}
\end{aligned}$$

Now using Theorem 5.3, we observe that for any $i \geq 3$, $\mathrm{Ind}(C_n^i) = \Sigma(\mathrm{Ind}(C_n^{i-2}))$ and therefore by induction on i, the result follows from $\mathrm{Ind}(C_n^1)$ and $\mathrm{Ind}(C_n^2)$. $\square$

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References

1. R. Aharoni, E. Berger, and R. Ziv, *Independent systems of representatives in weighted graphs*, Combinatorica **27** (2007), no. 3, 253–267. MR 2345810
2. E. Babson and D. Kozlov, *Proof of the Lovász conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 965–1007. MR 2335799
3. J. A. Barmak, *Star clusters in independence complexes of graphs*, Adv. Math. **241** (2013), 33–57. MR 3053703
4. B. Braun, *Independence complexes of stable Kneser graphs*, Electron. J. Combin. **18** (2011), no. 1, Paper 118, 17. MR 2811087
5. G. E. Bredon, *Topology and geometry*, vol. 139, Springer Science & Business Media, 2013.
6. R. Ehrenborg and G. Hetyei, *The topology of the independence complex*, European J. Combin. **27** (2006), no. 6, 906–923. MR 2226426
7. A. Engström, *Independence complexes of claw-free graphs*, European J. Combin. **29** (2008), no. 1, 234–241. MR 2368630
8. ______, *A local criterion for Tverberg graphs*, Combinatorica **31** (2011), no. 3, 321–332. MR 2835296

9. R. Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), no. 1, 90–145. MR 1612391
10. A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
11. K. Kawamura, *Homotopy types of independence complexes of forests*, Contrib. Discrete Math. **5** (2010), no. 2, 67–75. MR 2791291
12. ———, *Independence complexes of chordal graphs*, Discrete Math. **310** (2010), no. 15-16, 2204–2211. MR 2651818
13. D. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A **88** (1999), no. 1, 112–122. MR 1713484
14. ———, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, Berlin, 2008. MR 2361455
15. R. Meshulam, *Domination numbers and homology*, J. Combin. Theory Ser. A **102** (2003), no. 2, 321–330. MR 1979537
16. N. Nilakantan and S. Shukla, *Homotopy type of the independence complexes of a family of regular bipartite graphs*, arXiv e-prints (2017), arXiv:1709.04789.
17. J. Skwarski, *Operations on a graph G that shift the homology of the independence complex of G*, Skolan för teknikvetenskap, Kungliga Tekniska högskolan, 2010.

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