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SPIRALS IN PERIODIC TILINGS

JESSA CAMILLE DUERO*, BERNHARD KLAASSEN, AND ROVIN SANTOS

Abstract. Spiral tilings, as appealing as they are for their aesthetics, have not been studied well mathematically. One of the difficulties in this area of tiling theory is providing a mathematical definition of spiral tilings. A recently published attempt at providing a formal definition distinguishes a so-called L-spiral tiling (L-tiling) and an S-spiral tiling (S-tiling), with the two types being characterized by special properties of tile set partitions. Based on these existing definitions, we investigate the spiral structure in periodic tilings. Unlike spiral tilings, periodic tilings lend themselves easily to a definition and have been well studied. We first prove that it is not possible for periodic tilings to be S-tilings. We then study a subset of periodic tilings that can be L-tilings. In particular, we demonstrate that there exist examples for each type of isohedral tilings (a subset of periodic monohedral tilings) that are L-spirable.

1. Introduction

Spirals have been known since antiquity; they are a prominent feature in nature as well as in the arts. However, of the numerous studies done on spirals, only a few of them considered spirals as tilings rather than curves. A possible reason for this could be the difficulty in providing a mathematical definition for spiral tilings. This study takes advantage of a recently formulated definition to investigate spiral structures in periodic tilings. Before we proceed, we first clarify some basic terms that are used in this paper.

A plane *tiling* $\mathcal{T}$ (or *tessellation*) is a countable family of closed sets, called *tiles*, that cover the plane without gaps or overlaps of nonzero area. If the intersection of three or more tiles is nonempty, then this intersection is called a *vertex* of $\mathcal{T}$. An *edge* is (part of) the intersection of two tiles that connects two distinct vertices. All tiles considered in this research are closed topological disks and all tilings are k-hedral tilings, that is, tilings in which every tile is congruent to one of k different prototiles [3]. If $k = 1$, we call the tiling a *monohedral* tiling. Note that all the tiles in a k-hedral tiling are

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uniformly bounded. By uniformly bounded, we mean that there exist $r > 0$ and $R > 0$ such that it is possible for each tile in the tiling to completely contain a circle of radius r and be completely contained in a circle of radius R [3].

For a given plane tiling $\mathscr{T}$, one can look at the different isometries that leave $\mathscr{T}$ unchanged. These isometries are called *symmetries of* $\mathscr{T}$ and the group of all symmetries of $\mathscr{T}$ is called the *symmetry group of* $\mathscr{T}$ denoted by $\mathscr{S}(\mathscr{T})$. If the symmetry group $\mathscr{S}(\mathscr{T})$ contains two linearly independent translations, the tiling $\mathscr{T}$ is said to be *periodic*. Translations, rotations, and their compositions are called *direct isometries* [3].

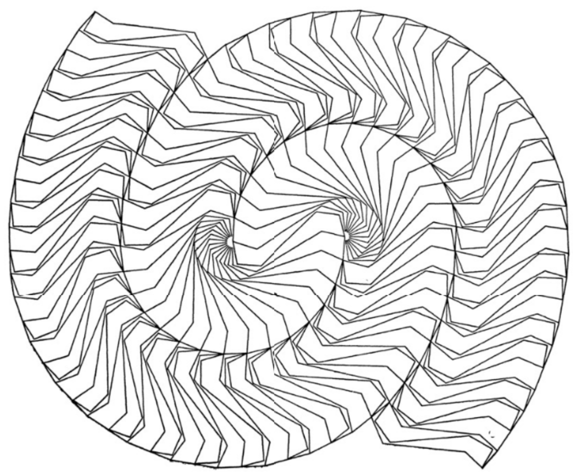


FIGURE 1. The Voderberg spiral [6].

The aim of this research is to examine the existence of spiral structures in periodic tilings. Spiral tilings were first studied with the discovery of the Voderberg Spiral in 1936. In 1987, Grünbaum and Shephard first attempted to define a spiral tiling as a monohedral tiling with a prototile T for which it is possible to mark T by one or several arcs so that the union of the corresponding arcs on all the tiles in the tiling consists of a finite number of unbounded simple curves ([3], see exercises of section 9.5). They admitted that their definition had several drawbacks, primarily its inability to capture the psychological aspect of spiral tilings. Since then, there have only been a relatively small number of papers published on this topic. As of this writing, the latest development in this field is a study [4] published in 2017; the succeeding mathematical definitions of spiral tilings are from that paper[1]. During the formulation of these definitions, the second author

[1]The second author posted a preprint which can be accessed in the following link https://arxiv.org/abs/2106.02827 in which, among other results, a further refinement of definitions from 2017 is discussed.

discussed them extensively with Branko Grünbaum. In the course of this correspondence, Grünbaum posed several mathematical questions, such as which general types of tilings would allow (or disallow) partitions in the sense of these definitions. This paper contains the first steps towards answering these questions.

Definition L. A partition of a plane tiling into two or more separate classes, called **arms**, is called a **spiral-like partition** or **L-partition** under the following conditions. (The plane is identified with the complex plane $\mathbb{C}$, where the origin is represented by a selected point of the tiling.)

L1: For each arm A (as a union of tiles from one class), there exists a curve $\theta : [0,\infty) \longrightarrow A \subset \mathbb{C}$ with $\theta(t) = r(t)\exp(\mathrm{i}\phi(\mathrm{t}))$, called a **thread**, where both r and ϕ are continuous and unbounded and ϕ is monotone. The curve θ must not meet or cross itself or any thread from another arm of the tiling.

L2: For each tile T in A, the intersection of the interior of T with the image of θ is nonempty and connected.

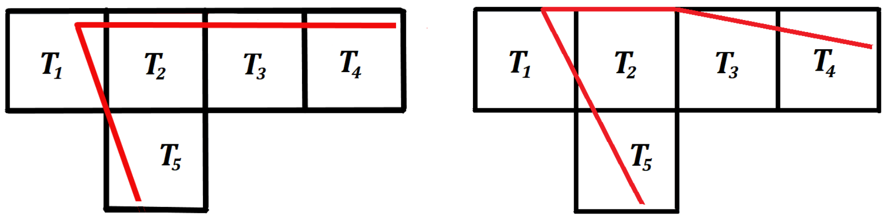
Let A be an arm of an L-partition. Two tiles $T_1, T_2 \in A$ are said to be **direct neighbors** if $T_1 \cap T_2$ is cut by the thread of A or contains more than a finite number of points. When $T_1 \cap T_2$ is a single point P, we say that $T_1 \cap T_2$ is "cut by a thread" if that thread intersects P. Figure 2 shows a few ways this can happen. In both Figures 2A and 2B, we have these pairs of direct neighbors in which each pair's intersection is an edge: T_1 and T_2, T_2 and T_3, and T_3 and T_4. The pair T_5 and T_1 in Figure 2A illustrates the case where $T_1 \cap T_5$ is a single point that is cut by a thread. The concept of direct neighbors is needed in proceeding from L-partitions to S-partitions, defined as follows:

Definition S. A partition of a plane tiling is defined as a **spiral partition** or **S-partition** under the following conditions.

S1: It is an L-partition.

S2: If any two tiles T_1, $T_2 \in A$ are direct neighbors and can be respectively mapped by a direct isometry τ onto another pair of tiles $\tau(T_1)$ and $\tau(T_2)$, these must also be direct neighbors within an arm. This rule can be ignored if the image pair contains the beginning of an arm, i.e., contains $\theta(0)$.

A plane tiling (with uniformly bounded tiles) for which it is possible to create an L-partition is *L-spirable* and an L-partitioned tiling is called an **L-tiling**. An L-tiling that satisfies S2 is called an **S-tiling**. Similarly, a tiling for which it is possible to create an S-partition is *S-spirable*. It can easily be verified that the Voderberg tiling (Figure 1) is S-spirable with two arms. More examples of S-tilings can be found in [1, 4, 5] and in Brian Wichmann's collection at www.tilingsearch.org/tree/t22.htm. In Wichmann's collection, none of the definitions in [4] apply to the tiling data129/F19 since the indicated spiral arms are not path-connected.



(A) (B)

FIGURE 2. Some ways a thread may run through a set of tiles.

Note that every arm in an S-tiling consists of tiles that are joined together in a way that creates a path through which a thread must run to meet every single tile of the arm. Whenever a thread runs from one tile to another, these two tiles are automatically direct neighbors by definition. Thus, a thread naturally defines a sequence of tiles in which the order is determined by the said path and every succeeding pair of tiles are direct neighbors. This observation is stated in the following remark.

Remark: Suppose θ^* is a nonempty connected subset of a thread[2] that intersects finitely many tiles in an arm A. Then one can always construct a sequence of tiles T containing all the tiles in A that are intersected by θ^* such that each pair of successive tiles in T are direct neighbors.

Proof. Define M^* to be the nonempty finite set of tiles in A intersected by θ^*. The existence of such a set is guaranteed since all tiles considered are uniformly bounded and θ^* is nonempty.

One can construct a sequence of tiles T containing all the tiles in M^*, where the order of the tiles is determined by the order in which the thread intersects the tiles in M^*. If there exists a point $\theta(p)$ on θ^* in which it meets two or more tiles at the same time, then sequence T may not be unique. Suppose $T_1, \dots, T_n$ are the tiles that are met by θ^* at $\theta(p)$, then:

(i) If some of these tiles do not contain $\theta(p \pm \varepsilon)$ for any arbitrarily small $\varepsilon > 0$ (see Figure 3A), then they can follow an arbitrary order in T since they all are direct neighbors.

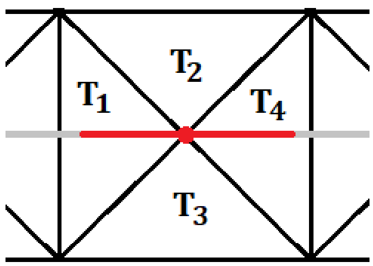
(ii) If for some arbitrarily small ε, $\theta(p+\varepsilon)$ lies on the edge $T_i \cap T_j$, then the order of these two tiles in T depends on the next tile intersected by θ^*. If the next tile is a direct neighbor of only one of those tiles, say T_j, then T_i must precede T_j in T (see Figure 3B). If both are

[2]In the following, "subset of a thread" means "subset of a thread's image" in the sense of a restriction to an interval of its domain.

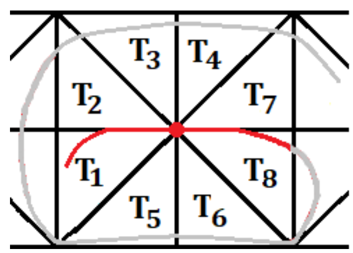
direct neighbors of the next tile, then T_i and T_j may appear in any order.

□

To further illustrate this remark, consider Figures 2A and 2B again. If θ^* is the red curve in each of the figures, then M^* consists of tiles T_1 to T_5. In Figure 2A, T_1, T_2, and T_5 are met by θ^* at the same point $\theta(p)$. However, among these three tiles, only T_2 does not contain $\theta(p \pm \varepsilon)$, for any arbitrarily small $\varepsilon > 0$. This means that T_2 should appear between T_5 and T_1 in the sequence. Assuming θ^* runs clockwise through the tiles of M^*, the sequence T described in remark 1.1 is the same for Figures 2A and 2B, which is $T_5, T_2, T_1, T_2, T_3, T_4$. Note that all successive pairs of tiles in this sequence are direct neighbors.



(A)



(B)

FIGURE 3. Let θ^* be the red curve that runs from left to right. If $\bigcap_{i=1}^{4} T_i = \theta(p)$ in (A), then the sequence T is either T_1, T_2, T_3, T_4 or T_1, T_3, T_2, T_4 (case i). In (B), if $\theta^* \cap (T_7 \cap T_8)$ contains more than a single point, then in the sequence T, tiles T_3, T_4, T_5 and T_6 can appear in any order (case i), followed by T_7 and T_8 (case ii).

2. S-SPIRABILITY

In this section, we investigate whether or not a periodic tiling can be an S-tiling. The defining property of periodic tilings is that their symmetry groups have linearly independent translations and this particular property is our main tool in determining if periodic tilings can be S-tilings. Although it is not forbidden by Definition L or S that tilings possess infinitely many spiral arms, we restrict ourselves to spirals with a finite number of arms, noting that the existence of spiral tilings with infinitely many arms remains an open question [5].

Let A be a spiral arm with thread θ_A. The tile that contains $\theta_A(0)$ is called the *beginning tile of* A. In an S-tiling $\mathcal{T}$, each arm is separated from the other arms by curves that are called *arm boundaries.* The arm boundaries are the topological boundaries of the spiral arms, thus separating the unions of the tiles in each arm. The thread θ_A, which is a continuous and unbounded curve, can meet these arm boundaries but cannot cross them. Hence, the arm boundaries of A are also continuous and unbounded curves.

Lemma 2.1. *Let T_1 and T_2 be tiles in an S-tiling, neither of which is a beginning tile. If $T_1 \cap T_2$ is an edge that lies on an arm boundary and $T_1 \cup T_2$ can be mapped by a direct isometry onto another pair of tiles, then the intersection of the tiles' images is also an edge that lies on an arm boundary.*

Proof. Suppose T_1 and T_2 are direct neighbors and can be mapped by a direct isometry onto another pair, say T_3 and T_4. Hence, $T_3 \cap T_4$ is also an edge. Suppose that T_3 and T_4 belong to the same arm, which means that these tiles must be direct neighbors (by definition). This contradicts Condition S2 since T_1 and T_2 (to which they could be mapped back) are not direct neighbors. Consequently, T_3 and T_4 belong to different arms and share an edge, which must then lie on an arm boundary. □

Let $\mathcal{T}$ be an S-tiling. Fix a circle $C_{\mathcal{T}}$ such that all beginning tiles in $\mathcal{T}$ lie in the interior of $C_{\mathcal{T}}$. This is possible since we are only considering an S-tiling with a finite number of arms.

Lemma 2.2. *Let A be an arm of an S-tiling $\mathcal{T}$ with symmetry group $\mathcal{S}(\mathcal{T})$. Suppose that θ^* is a connected subset of θ_A and M^* is the set of tiles in A that are intersected by θ^*. If both M^* and $\tau(M^*)$ lie outside the circle $C_{\mathcal{T}}$ (as defined above), where τ is a direct isometry in $\mathcal{S}(\mathcal{T})$, then $\tau(M^*)$ is also a subset of an arm in $\mathcal{T}$.*

Proof. Since M^* contains all tiles in A intersected by the connected path θ^*, then by remark 1.1, we can construct a sequence containing all the tiles in M^* in which each pair of successive tiles are direct neighbors. Each of these pairs is mapped (via τ) to another pair of tiles that must also be direct neighbors. The set $\tau(M^*)$ must therefore be a subset of a single arm. □

The set containing the edges of arm boundaries that are disjoint with any beginning tile shall be called *true arm boundaries*. This means that the edges in the arm boundaries that lie outside $C_{\mathcal{T}}$ are part of true arm boundaries. True arm boundaries still have the properties of arm boundaries, but an additional property holds.

Corollary 2.3. *For a set of edges in an S-tiling $\mathcal{T}$, the property of belonging to a true arm boundary is invariant under a direct isometry, provided that the image lies outside of a circle $C_{\mathcal{T}}$ which contains all arms' beginning tiles.*

This corollary follows directly from Lemmas 2.1, and the definition of the term true arm boundary.

We are now in a position to prove one of the main results of this paper. Note that the direct isometries mentioned in Condition S2 are not necessarily elements of the tiling's symmetry group. However, since periodic tilings, by definition, have translations in their symmetry group, we can use these translations to show that no S-partition can be induced from periodic tilings.

Proposition 2.4. *S-tilings with finitely many arms cannot be periodic.*

Proof. The idea of the proof is the following: Consider a 360° turn of (a part of) a spiral arm and translate it according to the assumed tiling's periodicity. We already know that this image is also a part of a spiral arm. However, it is impossible for the thread that is running through the image to be monotonic (with respect to the thread's angular coordinate), provided that this image is far enough from the original region. We now present the technical details.

Let $\mathcal{T}$ be a periodic S-tiling with only finitely many arms and let θ_A be a thread of an arm A in $\mathcal{T}$. Draw a straight line l passing through the origin of the complex plane that is parallel to a (nontrivial) translational symmetry t in $\mathcal{S}(\mathcal{T})$, the symmetry group of $\mathcal{T}$. Since $\mathcal{T}$ has a finite number of arms,

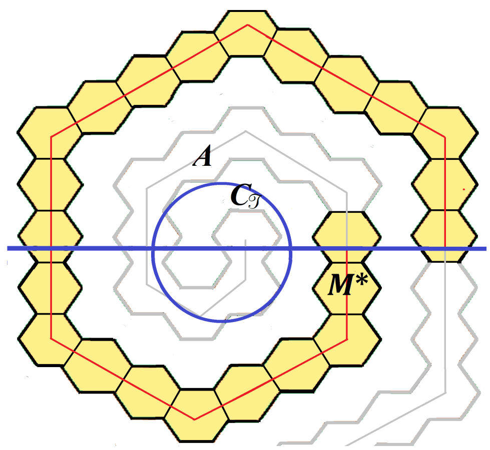


FIGURE 4

there exists a circle $C_{\mathcal{T}}$ such that all beginning tiles lie inside $C_{\mathcal{T}}$. Let us choose θ^* as a part of θ_A running 360° from line l to line l such that (i) M^*, the set of tiles in A that are intersected by θ^*, is outside $C_{\mathcal{T}}$; and (ii) region M^* is large enough such that there are at least two tiles in M^* fully lying on each side of l within the 360° turn. Here, we use the fact that the tiles are uniformly bounded. (In Figure 4, θ_A and l are represented by the red-gray curve and the blue straight line, respectively. The region $M^* \subset A$ is the yellow shaded region, which is outside $C_{\mathcal{T}}$ and θ^* is the red curve revolving 360° from line l to line l.)

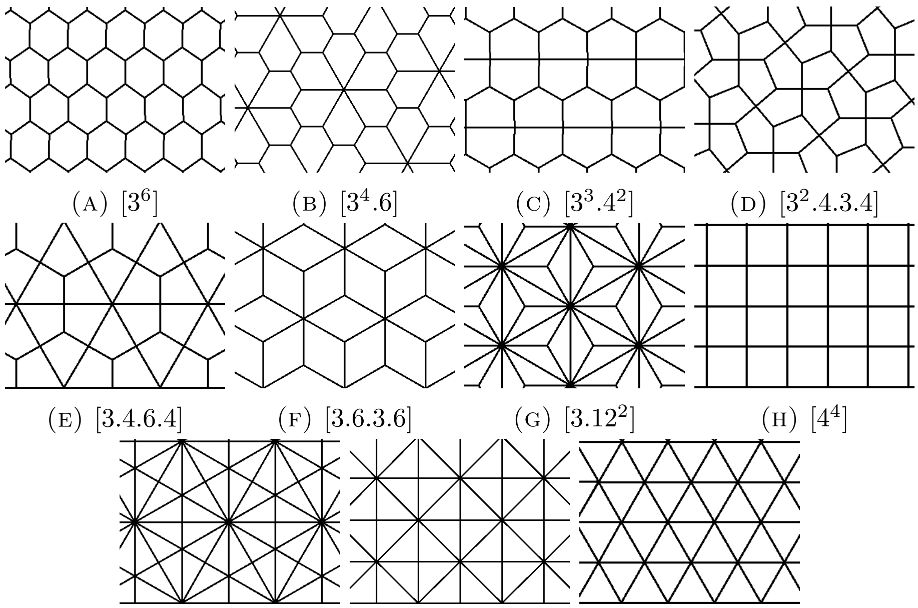
Note that the arm boundaries in M^* are part of true arm boundaries. By Corollary 2.3, true arm boundaries are mapped (via a direct isometry) to true arm boundaries provided that both regions lie outside of $C_{\mathcal{T}}$. Now, since the tiling is infinite, we can take $m \in \mathbb{N}$ to be large enough such that M^* and $t^m(M^*)$ are both outside $C_{\mathcal{T}}$. By Lemma 2.2, $t^m(M^*)$ is also part of an arm, say A', with a corresponding thread $\theta_{A'}$. By Condition L1, a thread must be monotonic in its angular coordinate. The thread $\theta_{A'}$ may not coincide with $t^m(\theta_A)$, but $\theta_{A'}$ must reach the interior of all of the tiles in the image region $t^m(M^*)$. Hence, it has to enter $t^m(M^*)$ at line l and afterward must be found on both sides of line l, which means that it cannot be monotonic with respect to the rotational angle. This results in a contradiction and completes the proof. □

3. L-spirability of isohedral tilings

We devote the rest of the paper to investigating whether some periodic tilings can satisfy Definition L. In particular, we explore the L-spirability of isohedral tilings (a subset of monohedral periodic tilings) and show each type is L-spirable. We have seen in the previous section that periodic tilings can never be S-tilings. However, the questions of whether every monohedral (or every periodic) tiling admits an L-partition and under which conditions periodic tilings can be L-tilings [4] remain open.

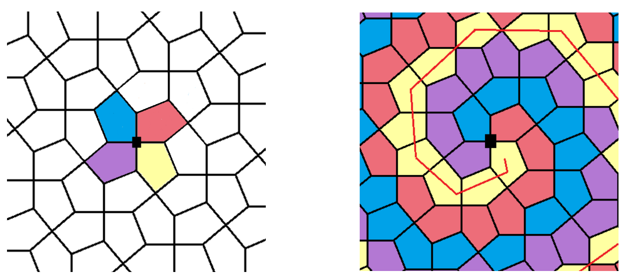
In a tiling $\mathcal{T}$, the *transitivity class* of a tile $T \in \mathcal{T}$ is the collection of all tiles in $\mathcal{T}$ to which T can be mapped by an element of the tiling's symmetry group $\mathcal{S}(\mathcal{T})$. $\mathcal{T}$ is said to be *isohedral* if all tiles in $\mathcal{T}$ belong to the same transitivity class. There are 81 types of isohedral tilings in the Euclidean plane, as enumerated by Grünbaum and Shephard in [2], where an *IH number* is used to classify each isohedral tiling. Monohedral tilings with regular vertices are also isohedral. A vertex is said the be a *regular vertex* if whenever v edges meet at a vertex of a tiling, then the angle between each consecutive pair of edges is $2\pi/v$. These monohedral tilings with regular vertices are called *Laves tilings* [2]. If the prototile of a Laves tiling is an n-gon whose vertices are of valence $v_1, v_2, \ldots, v_n$, then the Laves tiling is denoted by $[v_1.v_2.\cdots.v_n]$ (see Figure 5).

In this section, we first show that the 11 Laves tilings are L-spirable by creating L-partitions for each Laves tiling as shown in Table 1. These L-partitions are chosen based on the symmetry group of each Laves tiling. In particular, L-partitions are formed by considering an n-fold rotational symmetry in the tiling's symmetry group and then creating n arms that are transitive under the chosen rotational symmetry. For instance, a four-armed L-tiling with a four-fold rotational symmetry can be formed from $[3^2.4.3.4]$, a Laves tiling whose symmetry group is of type $p4g$. The four tiles joined at a vertex (that is a center of a four-fold rotation) are beginning tiles and form an orbit under the four-fold rotation centered at the said vertex. One can form the spiral structure in a clockwise or counterclockwise manner. Figure 6 illustrates the former. Each tile of the tiling is assigned to an arm so that the arms are invariant under the described (clockwise) four-fold rotation. In the case where the center of an n-fold rotation lies in the center of a tile T, it needs to be divided into n congruent parts to create an L-tiling with an n-fold rotational symmetry. However, the resulting tiling is no longer the original tiling. Take note that one can create other n-armed L-tilings that still preserve some rotational symmetries using different partitions.



(A) $[3^6]$ (B) $[3^4.6]$ (C) $[3^3.4^2]$ (D) $[3^2.4.3.4]$ (E) $[3.4.6.4]$ (F) $[3.6.3.6]$ (G) $[3.12^2]$ (H) $[4^4]$ (I) $[4.6.12]$ (J) $[4.8^2]$ (K) $[6^3]$

FIGURE 5. The 11 Laves tilings.



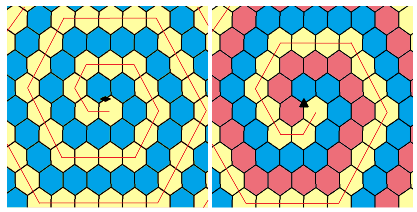
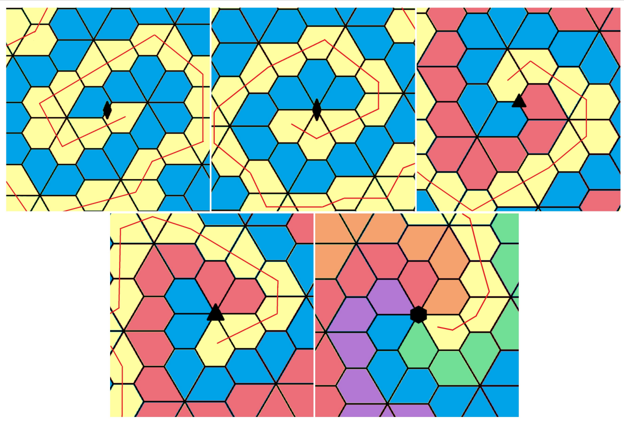
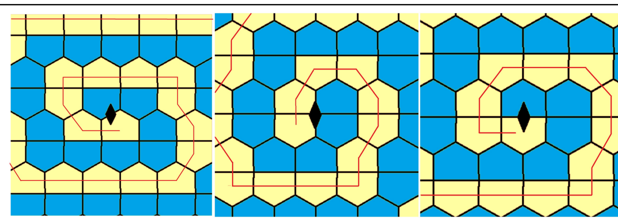
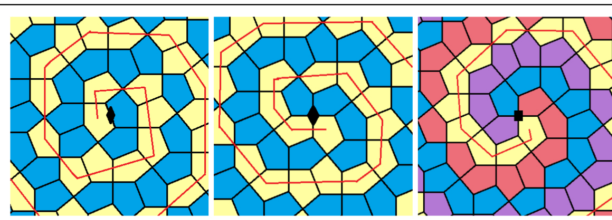
(A) (B)

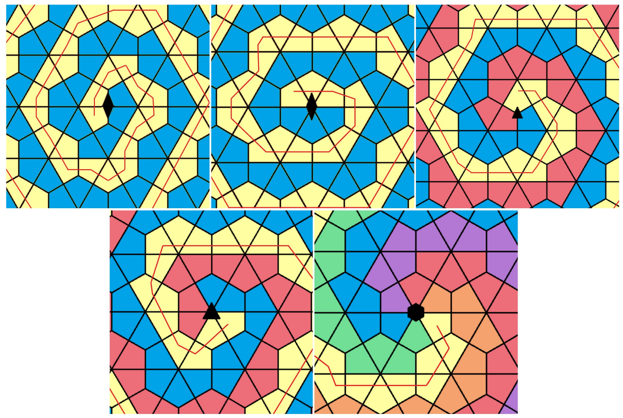
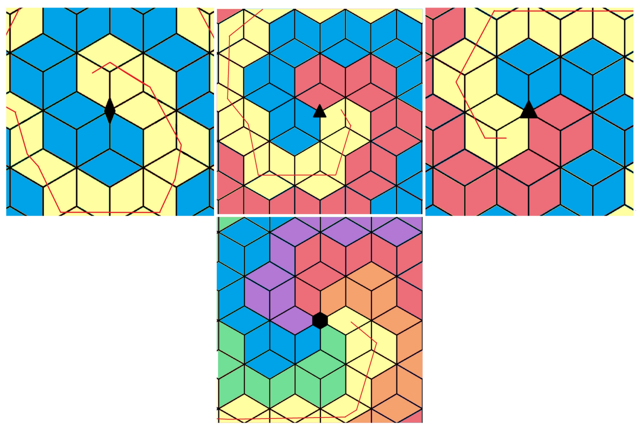
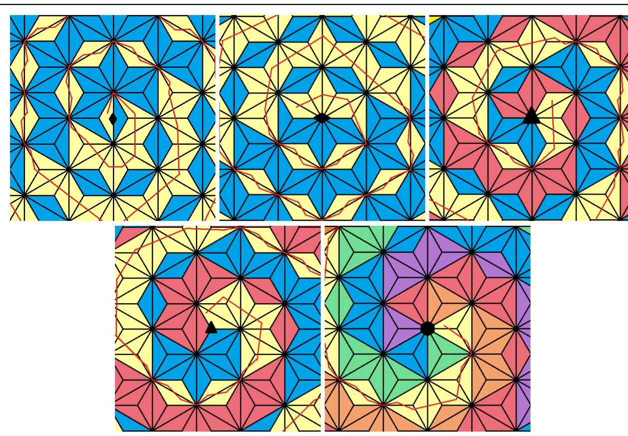
FIGURE 6. Creating a four-armed L-tiling for $[3^2.4.3.4]$.

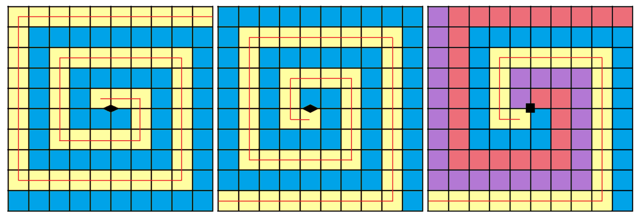
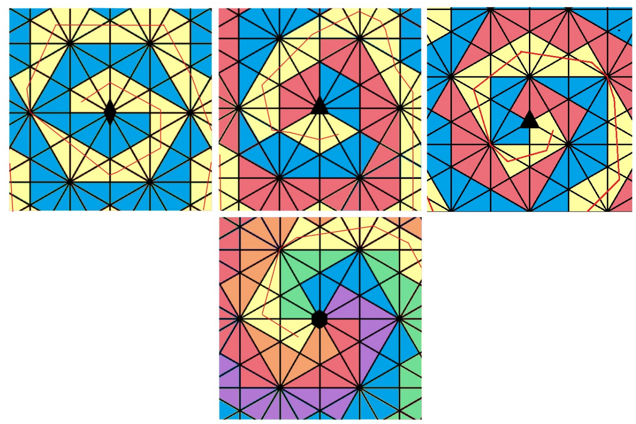
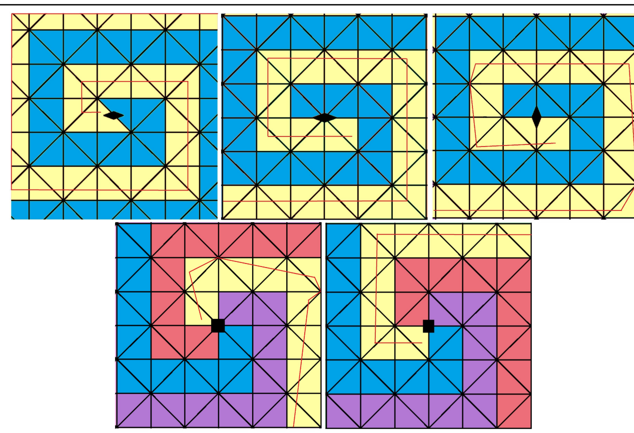
As we can see in Table 1, the spiral is more prominent if the tiles in an arm are connected at the edges all throughout the arm since one can more easily see how the arms partition the tilings into spirals. The red curve in each L-tiling in the table is the corresponding thread for the yellow arm. For the threads of the other arms, one may rotate this red curve under the rotational symmetries under which the arms are invariant. Note that L-partitions of the Laves tilings are not limited to the partitions shown in the table.

Table 1 demonstrates the L-spirability of the Laves tilings. In each of the arms in a partition lies a thread by Definition L. Additionally, we are able to establish the following result:

Proposition 3.1. *Let $\mathcal{T}$ be a Laves tiling. If an n-fold rotational symmetry is centered at a vertex of $\mathcal{T}$ or at the midpoint of an edge of $\mathcal{T}$, then $\mathcal{T}$ can be L-partitioned with n arms such that the resulting spiral-like tiling has symmetry group C_n.*

Laves tiling	Partitions
$[3^6]$ (IH1-IH20)	
$[3^4.6]$ (IH21)	
$[3^3.4^2]$ (IH22-IH26)	
$[3^2.4.3.4]$ (IH27-IH29)	

[3.4.6.4] (IH30-IH32)	
[3.6.3.6] (IH33-IH37)	
$[3.12^2]$ (IH38-IH40)	

$[4^4]$ (IH41-IH76)	
[4.6.12] (IH77)	
$[4.8^2]$ (IH78-IH82)	

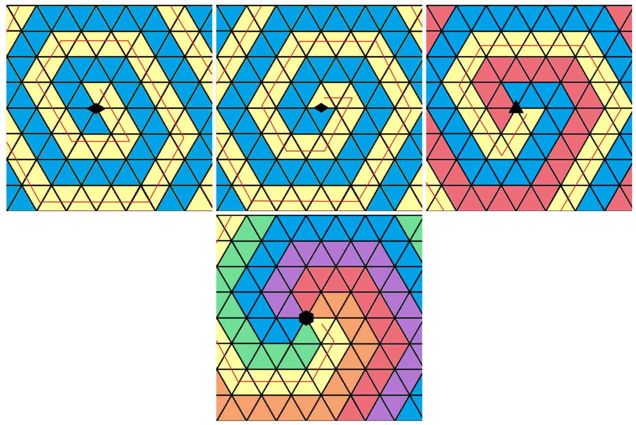
$[6^3]$ (IH83-IH93)	

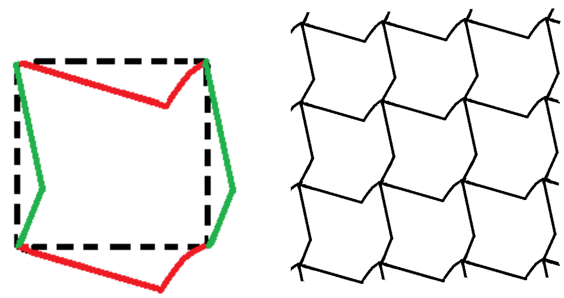
Table 1: L-partitions of Laves tilings with rotational symmetries.

Now, we proceed with our goal of showing that there are L-spirable examples for each type of isohedral tiling. Up to topological equivalence, the 81 types of isohedral tilings in the Euclidean plane can be grouped into 11 families and every family is represented by a Laves tiling. Thus, each isohedral tiling corresponds to the Laves tiling to which it is topologically equivalent.

Grünbaum and Shephard introduced the concept of *incidence symbols* and *adjacency diagrams* in [3] to characterize and enumerate all types of isohedral tilings. Incidence symbols indicate the symmetries in each prototile and how these prototiles make up the tiling while adjacency diagrams represent incidence symbols visually by utilizing the prototiles of the Laves tilings[3]. The authors of [3] illustrated the adjacency diagram of each type of isohedral tiling by marking the edges of the prototile of the Laves tiling to which it is topologically equivalent. These marked edges allow edge reshaping to come up with the IH type one wants to generate. In this paper, we modify one or several edges of a prototile of a Laves tiling into C-curves, J-curves, or S-curves (not to be confused with S-tilings) depending on the symmetries of the new prototile as done in [3] (see page 279). These modified curves can be made arbitrarily close to the original straight edges such that no curved edge is allowed which would increase the number of intersections between edges and threads compared to the L-partitioned Laves tiling. In this way, we made sure that we can produce isohedral tilings of every type that can be L-partitioned in the same manner, using the same threads, as the corresponding Laves tiling in Table 1. While we are not guaranteed that

[3]The reader may also refer to https://www.jaapsch.net/tilings/mclean/index.html in which the author used an alternative approach to uniquely characterize each type of isohedral tilings in a visually informative way.

all isohedral tilings can be made into L-tilings, these L-partitions created from Laves tilings are sufficient to conclude that examples for each type of isohedral tiling exist that are L-spirable.



(A) (B)

FIGURE 7. Tiling $\mathcal{T}$ of type IH41 constructed from $[4^4]$.

For example, the isohedral tiling $\mathcal{T}$ in Figure 7B of type IH41 can be created from $[4^4]$ (Figure 5H). As we can see, the two pairs of opposite edges of the square prototile in $[4^4]$ are reshaped into J-curves in line with the characterization of IH41 in [3]. From this characterization, the opposite edges are images of one another under some translations in $\mathcal{S}(\mathcal{T})$, as illustrated by the pair of red edges and the pair of green edges in Figure 7A. In this example, the adjusted edges do not increase the number of intersections between edges and threads compared to the L-partitioned Laves tiling.

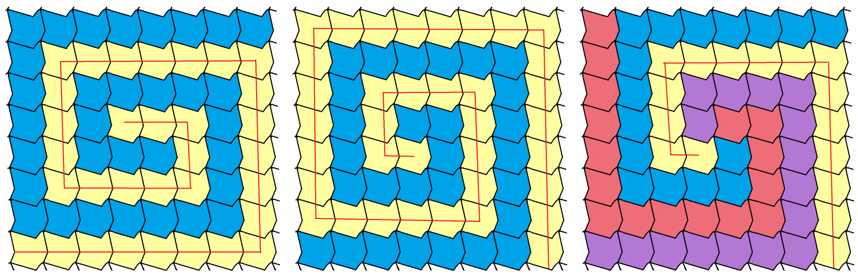
We see in Figure 8 the L-partitions of $\mathcal{T}$ in accordance with the L-tilings obtained from $[4^4]$. Clearly, all these L-tilings have trivial rotational symmetries even if the corresponding L-tilings for $[4^4]$ have two- and four-fold rotational symmetries. The rotational symmetries found in the L-partitions of a Laves tiling may be lost when the same partitions are used in the isohedral tilings from the family represented by the Laves tiling. This is certain to happen if the symmetry group of the generated isohedral tiling is of type $p1$ (as is the case for $\mathcal{T}$), pg, pm, and cm.

The example shown above illustrates that one can always generate at least one example from each of the 81 types of isohedral tilings such that each generated tiling adapts the thread of its corresponding Laves tiling. In this way, these generated isohedral tilings can be partitioned into L-tilings in the same way as their corresponding Laves tiling, in line with Table 1.

Remark: One can modify the edges of a Laves tiling to construct an L-tiling for at least one example of each type of isohedral tiling using the same partition as that of the corresponding Laves tiling according to Table 1.

4. Discussion and conclusion

The motivation behind the definitions from [4] for spiral tilings is to capture, with mathematical rigor, the psychological effect alluded to by



(A) (B) (C)

Figure 8. L-partitions of $\mathcal{T}$ in line with the partitions of $[4^4]$ in Table 1.

Grünbaum and Shephard in their 1987 work. Intuitively, we can see that there are no inherent spiral structures in any periodic tiling. In this work, we have provided mathematical support, using these definitions, for that intuitive result.

If a periodic tiling is S-spirable with finite number of arms, then the S-partitioning of the arms can be translated through the (at least two) linearly independent translations in $\mathcal{S}(\mathcal{T})$ which, intuitively, should not be possible in a spiral tiling. In fact, the proof of Proposition 2.4 holds even when the tiling has only one translational symmetry in its symmetry group. Hence, only tilings with no translational symmetry can be S-tilings.

This work is the first step towards answering Branko Grünbaum's question about general properties of tilings that may admit L-partitions which was already mentioned as an open problem in [4]. Although we have not yet completely characterized the L-spirability of monohedral (or periodic) tilings, we have shown that we can find at least one example of an L-spirable tiling for each of the 81 types of isohedral tilings. In fact, we conjecture that all periodic monohedral tilings are L-spirable.

The authors are working on a separate paper to demonstrate the existence of periodic dihedral tilings that are not L-spirable. For now, we propose that not all periodic tilings are L-spirable. Moreover, one-armed spirals are also introduced in [4] and we present proof that one-armed spiral tilings (according to Definition O in [4]) cannot be periodic in a separate paper.

5. Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest. Moreover, this research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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University of the Philippines Los Baños
E-mail address: jcduero@up.edu.ph

Fraunhofer Institute SCAI, Sankt Augustin, Germany
E-mail address: bernhard.klaassen@scai.fraunhofer.de

University of the Philippines Diliman
E-mail address: rbsantos8@up.edu.ph