Contributions to Discrete Mathematics

Volume 17, Number 2, Pages 23–40
ISSN 1715-0868

COVERINGS WITH CONGRUENT AND NON-CONGRUENT HYPERBALLS GENERATED BY DOUBLY TRUNCATED COXETER ORTHOSCHEMES

MIKLÓS EPER AND JENŐ SZIRMAI

Abstract. After the investigation of the congruent and non-congruent hyperball packings related to doubly truncated Coxeter orthoscheme tilings [Acta Univ. Sapientiae, Mathematica, **11**, 2 (2019), 437–459], we consider the corresponding covering problems. In *Non-fundamental trunc-simplex tilings and their optimal hyperball packings and coverings in hyperbolic space* the authors gave a partial classification of supergroups of some hyperbolic space groups whose fundamental domains will be integer parts of truncated tetrahedra, and determined the optimal congruent hyperball packing and covering configurations belonging to some of these classes.

In this paper, we complement these results with the investigation of the non-congruent covering cases and the remaining congruent cases. We prove, that between congruent and non-congruent hyperball coverings the thinnest belongs to the $\{7,3,7\}$ Coxeter tiling with density ≈ 1.26829. This covering density is smaller than the conjectured lower bound density of L. Fejes Tóth for coverings with balls and horoballs.

We also study the local packing arrangements related to $\{u,3,7\}$ ($6 < u < 7,\ u \in \mathbb{R}$) doubly truncated orthoschemes and the corresponding hyperball coverings. We prove, that these coverings are achieved their minimum density at parameter $u \approx 6.45953$ with covering density ≈ 1.26454 which is smaller than the above record-small density, but this hyperball arrangement related to this locally optimal covering can not be extended to the entire $\mathbb{H}^3$.

Moreover, we see that in the hyperbolic plane $\mathbb{H}^2$ the universal lower bound of the congruent circle, horocycle, hypercycle covering density $\sqrt{12}/\pi$ can be approximated arbitrarily well also with non-congruent hypercycle coverings generated by doubly truncated Coxeter orthoschemes.

Received by the editors May 18, 2021, and in revised form May 31, 2021.

2000 *Mathematics Subject Classification.* 52C17, 52C22, 52B15.

Key words and phrases. Hyperbolic geometry, hyperball packings and coverings, covering density, Coxeter tilings.

1. Introduction

The investigation of optimal dense packings and coverings with congruent balls in spaces with constant curvature is one of the important topics in discrete geometry. The most explored is the Euclidean case. One of the greatest results is the solution of the famous Kepler-conjecture [12] (Hilbert's 18th problem). Its computer-supported proof was given by Thomas Hales in the early 2000s, which is based on the ideas of László Fejes Tóth [9].

In the hyperbolic case, there are far more open questions. In n-dimension ($n \geq 3$) e.g. it isn't clear when the most dense packing is realised with classical balls. The so far known highest packing density is ≈ 0.77147 with classical balls in $\mathbb{H}^3$, published in [21], where the authors also determined a classical ball configuration, which provides the so far known thinnest covering density ≈ 1.36893.

Moreover, in hyperbolic space, the definition of the density of the packings and coverings is crucial, as was shown by Károly Böröczky in his works [4] and [5]. The most common density definition considers the local density of the balls related to their Dirichlet-Voronoi cells. We will use as well this local density definition and also its extension. A very important result of the classical ball and horoball packings is the following theorem:

Theorem 1.1 (K. Böröczky [6], [7])**.** *In an n-dimensional space of constant curvature, consider a packing of spheres of radius r. In the case of spherical space, assume that $r < \pi/4$. Then the density of each sphere in its Dirichlet–Voronoi cell cannot exceed the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centers.*

In hyperbolic space $\mathbb{H}^n$ ($n \geq 2$) in addition to the classical spheres, there are two other types of balls: horoballs and hyperballs, which are non-compact "balls", and the above packing and covering problems with these kinds of balls were also intensively investigated. The densest packing configuration, which Theorem 1.1 states, can be realized in $\mathbb{H}^3$, but surprisingly not with classical balls, but with horoballs, providing density ≈ 0.85328 (this density can be attained with more different types of horoball packings related to fully asymptotic Coxeter tiling [15]). In higher dimensions ($n = 4, ..., 9$) there are also interesting results with high densities (see [16], [17], [18]). It was also shown, that in $\mathbb{H}^n$ ($n \geq 4$) the density of locally optimal horoball packing related to n-dimensional fully asymptotic regular simplices exceeds the conjectured bound, for example in $\mathbb{H}^4$ it attains density ≈ 0.77147, but the corresponding configuration can not be extended to the entire hyperbolic space (see [23], [24]). Another question is, what will be the configuration in certain dimensions for optimal horoball packing and covering with horoballs of "different types" [19], [25].

In the epoch-making book of László Fejes Tóth [9] one can find the description of a horoball covering with density ≈ 1.280, which belongs to the $\{6, 3, 3\}$ tiling, and here conjectured that this would give the thinnest covering in the $\mathbb{H}^3$.

We know even less about the packings and coverings with hyperballs. In the hyperbolic plane ($n = 2$) I. Vermes proved, that the upper bound for *congruent hypercycle packing density* is $3/\pi$ [35], and the lower bound for *congruent hypercycle covering density* is $\sqrt{12}/\pi$ [36]. We note, that the densest hypercycle packings and least dense covering can be realized with the same density as the optimal packing and coverings using horospheres, as it was shown e.g. in [37]. However, there are no results related to the densest hypercycle packings and thinnest hypercycle coverings with noncongruent hypercycles.

Moreover, in higher dimensions, there are few results related to congruent and noncongruent hyperball packings and coverings. The locally optimal hyperball packing configuration was researched in previous works related to several tilings: to tilings with truncated regular hyperbolic simplices in [30], [32]; to tilings with cubes and octahedrons in [26] (that yields density ≈ 0.84931 in noncongruent cases); to tilings with truncated Coxeter orthoschemes in [22] (that yields density ≈ 0.81335).

But these are just some of the works, where the densest hyperball packings were investigated in $\mathbb{H}^n$ (see also [28], [27]). In [29] the least dense hyperball coverings are determined related to 3-,4- and 5-dimensional Coxeter tilings, which can be derived from Coxeter orthoschemes by truncating the principal vertex with its polar plane. In several cases, there were found locally optimal configurations, which densities exceeds the Böröczky-Florian upper bound related to the classical ball and horoball packings.

Other types of ball packings and coverings in $\mathbb{H}^n$ are the so-called hyphor packings [31] and coverings [8], where in a configuration we use both horoball and hyperball, and the fundamental domain of the tilings are simply truncated simply asymptotic Coxeter orthoschemes. In the hyperbolic plane in both cases (packing and covering) the theoretic bound of the density (as above) can be attained in limit. In $\mathbb{H}^3$ the optimal packing density is ≈ 0.83267, and the optimal covering density is ≈ 1.27297 both related to the $\{7, 3, 6\}$ Coxeter tiling. Moreover, we also considered configurations $\{p, 3, 6\}$ $(6 < p < 7,\ p \in \mathbb{R})$, and we got better densities: ≈ 0.85397 for packing, ≈ 1.26885 for covering, but these tilings can not be extended to the entire hyperbolic space.

In [33] we considered congruent hyperball packings in 3-dimensional hyperbolic space and developed a decomposition algorithm that for each saturated hyperball packing provides a decomposition of $\mathbb{H}^3$ into truncated tetrahedra. Therefore, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices. In [34] we proved, using the above results, that the density upper bound of the saturated congruent hyperball (hypersphere) packings related to the corresponding truncated tetrahedron cells is realized in regular truncated tetrahedra with density ≈ 0.86338. Furthermore, we prove that the density of locally optimal congruent hyperball arrangement in regular truncated tetrahedron is not monotonically increasing function of the height (radius) of corresponding optimal hyperball, contrary to the ball (sphere) and horoball (horosphere) packings.

Our discussion in this paper related to previous investigations (see [20]) where the authors considered some tilings generated by doubly truncated orthoschemes in $\mathbb{H}^3$, and determined their optimal packing and covering configurations with congruent hyperballs. In [20] was also shown, that the tiling $\{7,3,7\}$ provides ≈ 1.26829 covering density, which is the currently known smallest ball covering density in $\mathbb{H}^3$. We note here that in [22] the corresponding packing problem was solved for congruent and noncongruent hyperballs related to similar orthoschemes. The tiling $\{7,3,7\}$ provides the optimal ≈ 0.81335 packing density, which is realized with packings of congruent hyperballs.

In this paper, we complete and close the investigation started in the mentioned previous papers. Now, we prove, that the thinnest covering with congruent or noncongruent hyperballs related to doubly truncated Coxeter orthoschemes generated tilings is realized at the $\{7,3,7\}$ tiling with density ≈ 1.26829 (see also [20]). *We note here that this covering density is smaller than the conjectured lower bound of L. Fejes Tóth density for coverings with balls and horoballs.*

Moreover, we also consider $\{u,3,7\}$ $(6 < u < 7,\ u \in \mathbb{R})$ tilings, where the locally optimal density is ≈ 1.26454, at parameter $u \approx 6.45953$.

Finally, we discuss the hypercycle coverings with congruent or noncongruent hypercycles, related to doubly truncated Coxeter orthoschemes generated tilings in the hyperbolic plane $\mathbb{H}^2$. We prove that the universal lower bound of the congruent circle or horocycle and hypercycle covering density $\sqrt{12}/\pi$ can be approximated arbitrarily well with non-congruent hypercycle coverings.

2. Basic notions

2.1. **The projective model of hyperbolic space $\mathbb{H}^3$.** For the computations, we use the projective Beltrami-Cayley-Klein model of hyperbolic space. The model is defined in the $\mathbb{E}^{1,n}$ Lorentz space with signature $(1,n)$, i.e. consider $\mathbf{V}^{n+1}$ real vector space equipped with the bilinear form:

$$\langle\ \mathbf{x},\ \mathbf{y}\rangle = -x^0y^0 + x^1y^1 + \dots + x^ny^n.$$

In the vector space consider the following equivalence relation:

$$\mathbf{x}(x^0,\dots,x^n) \sim \mathbf{y}(y^0,\dots,y^n) \Leftrightarrow \exists c \in \mathbb{R}\backslash\{0\} : \mathbf{y} = c\cdot\mathbf{x}.$$

The factorization with $\sim$ induces the $\mathcal{P}^n(\mathbf{V}^{n+1},\boldsymbol{V}_{n+1})$ n-dimensional real projective space. In this space to interpret the points of $\mathbb{H}^n$ hyperbolic space, consider the following quadratic form:

$$Q = \{[\mathbf{x}] \in \mathcal{P}^n | \langle\ \mathbf{x},\ \mathbf{x}\rangle = 0\} =: \partial\mathbb{H}^n,$$

The inner points relative to the cone-component determined by Q are the points of $\mathbb{H}^n$ (for them $\langle\ \mathbf{x},\ \mathbf{x}\rangle < 0$), the point of $Q = \partial\mathbb{H}^n$ are called the points at infinity, and the points lying outside realtive to Q are outer points of $\mathbb{H}^n$ (for them $\langle \mathbf{x}, \mathbf{x}\rangle > 0$). We can also define a linear polarity between the points and hyperplanes of the space: the polar hyperplane of a point $[\mathbf{x}] \in \mathcal{P}^n$ is $\mathrm{Pol}(\mathbf{x}) := \{[\mathbf{y}] \in \mathcal{P}^n | \langle \mathbf{x}, \mathbf{y}\rangle = 0\}$, and hence $\mathbf{x} \in \mathbf{V}^{n+1}$ is incident with $\boldsymbol{a} \in \boldsymbol{V}_{n+1}$ if and only if $\langle \mathbf{x}, \boldsymbol{a}\rangle = 0$. In this projective model we can define a metric structure related to the above bilinear form, where for the distance of two proper points:

$$\cosh(d(\mathbf{x},\mathbf{y})) = \frac{-\langle \mathbf{x}, \mathbf{y}\rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle}}. \tag{2.1}$$

This corresponds to the distance formula in the well-known Beltrami-Cayley-Klein model.

2.2. **Hyperballs.** We implement the covering of $\mathbb{H}^n$ with hyperballs: we assign hyperballs to a doubly truncated orthoscheme, and if the hyperballs cover it then accordingly to the images of the orthoscheme, the images of the hyperballs provide a covering of the space.

The equidistant surface (or hypersphere) is a quadratic surface that lies at a constant distance from a plane in both halfspaces. The infinite body of the hypersphere is called a hyperball. The n-dimensional *half-hypersphere* ($n = 2, 3$) with distance h to a hyperplane π is denoted by $\mathcal{H}_n^h$. In the above model, the equation of $\mathcal{H}_n^h$ can be written:

$$1 - \sum_{i=1}^{n} x_i^2 = \left(-\frac{u_0}{\sinh h} + \sum_{i=1}^{n} \frac{u_i}{\sinh h} x_i \right)^2, \tag{2.2}$$

where $(u_0, ..., u_n)$ is the polar point of π hyperplane.

The volume of a bounded hyperball piece $\mathcal{H}_n^h(\mathcal{A}_{n-1})$ bounded by an $(n-1)$-polytope $\mathcal{A}_{n-1} \subset \pi$, $\mathcal{H}_n^h$ and by hyperplanes orthogonal to π derived from the facets of $\mathcal{A}_{n-1}$ can be determined by the following formulas that follow from the suitable extension of the classical method of J. Bolyai [2]:

$$Vol_3(\mathcal{H}_3^h(\mathcal{A}_2)) = \frac{1}{4} Vol_2(\mathcal{A}_2) \left[\sinh(2h) + 2h\right], \tag{2.3}$$

where the volume of the hyperbolic $(n-1)$-polytope $\mathcal{A}_{n-1}$ lying in the plane π is $Vol_{n-1}(\mathcal{A}_{n-1})$.

From the equation of the $\mathcal{H}_n^h$, we can see that the equivalents of the hyperballs in our model will be ellipsoids.

3. Covering with hyperballs in hyperbolic space $\mathbb{H}^3$

3.1. Coxeter orthoschemes and tilings.

Definition 3.1. *In the $\mathbb{H}^n$ ($2 \le n \in \mathbb{N}$) space a complete orthoscheme $\mathcal{O}$ of degree d ($0 \le d \le 2$) is a polytope bounded with hyperplanes $H^0, \dots, H^{n+d}$, for which $H^i \perp H^j$, unless $j \ne i-1, i, i+1$.*

In the classical ($d = 0$) case let denote the vertex opposite to H^i hyperplane with A_i ($0 \leq i \leq n$), and let denote the dihedral angle of H^i and H^j planes with α^{ij} (hence α^{ij} if $0 \leq i < j - 1 \leq n$).

In this paper, we deal with orthoschemes of degree $d = 2$, which can be described geometrically, as follows. We can give the sequence of the vertices of the orthoschemes $A_0, \dots, A_n$, where A_iA_{i+1} edge is perpendicular to $A_{i+2}A_{i+3}$ edge for all $i \in \{0, \dots, n-3\}$. Here A_0 and A_n are called the principal vertices of the orthoschemes. In the case $d = 2$ these principal vertices are outer points of the above model, so they are truncated by its polar planes $\mathrm{Pol}(A_0)$ and $\mathrm{Pol}(A_n)$, and the orthoscheme is called doubly truncated (see Fig. 1). Now, we suppose that the A_0A_n line intersects the model, and does not deal with the other case.

In general, the Coxeter orthoschemes were classified by H.-C. Im Hof, he proved that they exist in dimension ≤ 9, and gave a full list of them [10] [11].

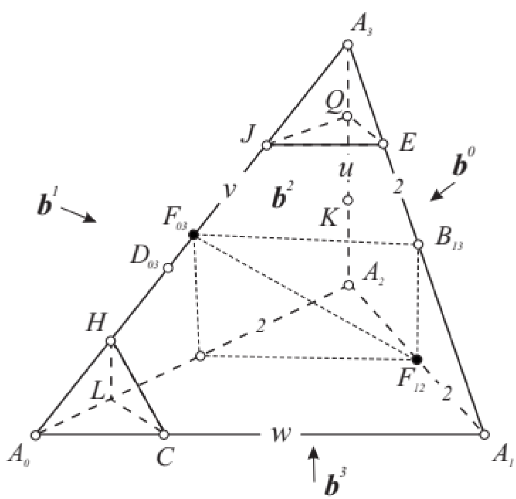


Figure 1. 3-dimensional Coxeter orthoscheme of degree $d = 2$ with outer vertices A_0, A_3, truncated by $\pi^0, \pi^3 = HLC, JQE$ polar planes.

Now consider the reflections on the facets of the doubly truncated orthoscheme, and denote them with $r_1, \dots, r_{n+3}$, hence define the group

$$\mathbf{G} = \langle r_1, \dots, r_{n+3} | (r_ir_j)^{m_{ij}} = 1 \rangle,$$

where $\alpha^{ij} = \pi/m_{ij}$, so $m_{ii} = 1$, and if $m_{ij} = \infty$ (i.e. H^i and H^j are parallel), than to the r_i, r_j pair belongs no relation. Suppose that $2 \leq m_{ij} \in \{\mathbb{N} \cup \infty\}$ if $i \neq j$. The Coxeter group G acts on hyperbolic space $\overline{\mathbb{H}}^n$ properly discontinously, thus the images of the orthoscheme under this action provide a $\mathcal{T}$ tiling of $\overline{\mathbb{H}}^n$ (i.e. the images of the orthoscheme fills the $\overline{\mathbb{H}}^n$ without overlap).

For the *complete Coxeter orthoschemes* $\mathcal{O} \subset \mathbb{H}^n$ we adopt the usual conventions and sometimes even use them in the Coxeter case: if two nodes are related by the weight $\cos(\pi/m_{ij})$ then they are joined by a $(m_{ij}-2)$-fold line for $m_{ij} = 3,\ 4$ and by a single line marked by m_{ij} for $m_{ij} \geq 5$. In the hyperbolic case if two bounding hyperplanes of $\mathcal{O}$ are parallel, then the corresponding nodes are joined by a line marked ∞. If they are divergent then their nodes are joined by a dotted line.

In the following, we concentrate only on dimension 3 and on hyperbolic Coxeter-Schläfli symbol of the complete orthoscheme tiling $\mathcal{P}$ generated by reflections on the facets of a complete orthoscheme $\mathcal{O}$. To every scheme there is a corresponding symmetric 4×4 matrix (b^{ij}) where $b^{ii} = 1$ and, for $i \neq j \in \{0,1,2,3\}$, b^{ij} equals to $-\cos\alpha_{ij}$ with all dihedral angles α_{ij} between the faces H_i, H_j of $\mathcal{O}$.

For example, (b^{ij}) in formula (3.1) is the so-called Coxeter-Schläfli matrix with parameters $(u;v;w)$, i.e. $\alpha_{01} = \pi/u$, $\alpha_{12} = \pi/v$, $\alpha_{23} = \pi/w$. Now only $3 \leq u,v,w$ come into account (see [10, 11]).

$$(b^{ij}) = \langle \boldsymbol{b^i}, \boldsymbol{b^j} \rangle := \begin{pmatrix} 1 & -\cos\frac{\pi}{u} & 0 & 0 \\ -\cos\frac{\pi}{u} & 1 & -\cos\frac{\pi}{v} & 0 \\ 0 & -\cos\frac{\pi}{v} & 1 & -\cos\frac{\pi}{w} \\ 0 & 0 & -\cos\frac{\pi}{w} & 1 \end{pmatrix}. \tag{3.1}$$

This 3-dimensional complete (truncated or frustum) orthoscheme $\mathcal{O} = \mathcal{O}(u,v,w)$ and its reflection group $\mathbf{G}_{uvw}$ will be described in Fig. 1, 2, and by the symmetric Coxeter-Schläfli matrix (b^{ij}) in formula (3.1), furthermore by its inverse matrix (h_{ij}) in formula (3.2).

(3.2)

$$(h_{ij}) = (b^{ij})^{-1} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle :=$$

$$= \frac{1}{B} \begin{pmatrix} \sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} & \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} \\ \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \sin^2\frac{\pi}{w} & \cos\frac{\pi}{v} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{v} & \sin^2\frac{\pi}{u} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} & \sin^2\frac{\pi}{u} - \cos^2\frac{\pi}{v} \end{pmatrix},$$

where

$$B = \det(b^{ij}) = \sin^2\frac{\pi}{u}\sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} < 0, \quad \text{i.e. } \sin\frac{\pi}{u}\sin\frac{\pi}{w} - \cos\frac{\pi}{v} < 0.$$

The volume of an doubly truncated $HLCA_1A_2JQE$ Coxeter orthoscheme with outer vertices A_0, A_3 (see Fig. 1) in $\mathbb{H}^n$ can be determined by the following theorem of R. Kellerhals [13, 14].

Theorem 3.2. *The volume of a three-dimensional hyperbolic complete orthoscheme (except Lambert cube cases) $\mathcal{S}$ is expressed with the essential angles $\alpha^{01}, \alpha^{12}, \alpha^{23}$, $(0 \leq \alpha^{ij} \leq \pi/2)$ in the following form:*

$$Vol_3(\mathcal{S}) = \frac{1}{4}\{\mathcal{L}(\alpha^{01}+\theta) - \mathcal{L}(\alpha^{01}-\theta) + \mathcal{L}(\frac{\pi}{2}+\alpha^{12}-\theta)+ \\ + \mathcal{L}(\frac{\pi}{2}-\alpha^{12}-\theta) + \mathcal{L}(\alpha^{23}+\theta) - \mathcal{L}(\alpha^{23}-\theta) + 2\mathcal{L}(\frac{\pi}{2}-\theta)\},$$

where $\theta \in [0, \frac{\pi}{2})$ is defined by the following formula:

$$\tan(\theta) = \frac{\sqrt{\cos^2\alpha^{12} - \sin^2\alpha^{01}\sin^2\alpha^{23}}}{\cos\alpha^{01}\cos\alpha^{23}}$$

and where

$$\mathcal{L}(x) := -\int_0^x \log|2\sin t| dt$$

denotes the Lobachevsky function.

3.2. **Hyperball coverings.** Consider in 3-dimensional hyperbolic space $\mathbb{H}^3$ a doubly truncated Coxeter orthoscheme, with Coxeter-Schläfli symbol $\{u, v, w\}$. Its Coxeter-Schläfli matrix (c^{ij}) and its inverse are described in (4) and (5).

Set the $A_0A_1A_2A_3$ orthoscheme in the model centered at $O = (1, 0, 0, 0)$, so the $A_k[\mathbf{a_k}]$ $(k = 1, 2)$ vertices are proper points, i.e. $h_{kk} = \langle \mathbf{a}_k, \mathbf{a}_k \rangle < 0$ if $(k = 1, 2)$, and we can check easily that it's always fulfilled. On the other hand $A_k[\mathbf{a_k}]$ $(k = 0, 3)$ principal vertices are outer points, i.e. $h_{kk} = \langle \mathbf{a}_k, \mathbf{a}_k \rangle > 0$ if $(k = 0, 3)$, and from this we get the conditions

$$\sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} < 0 \text{ and } \sin^2\frac{\pi}{u} - \cos^2\frac{\pi}{v} < 0,$$

or equivalently

$$\frac{1}{w} + \frac{1}{v} < \frac{1}{2} \text{ and } \frac{1}{u} + \frac{1}{v} < \frac{1}{2}.$$

It means, that we have the following infinite series of $\mathcal{F}_u^{(v,w)}$ Coxeter orthoschemes with two outer vertices (see [21] for details):

- $\{u, v, w\}$, where $u \geq 3$, $v \geq 7$ and $w \geq 3$,
- $\{u, v, w\}$, where $u \geq 4$, $v = 5, 6$ and $w \geq 4$,
- $\{u, v, w\}$, where $u \geq 5$, $v = 4$ and $w \geq 5$,
- $\{u, v, w\}$, where $u \geq 7$, $v = 3$ and $w \geq 7$.

We truncate the orthoscheme with the polar planes $\pi^0 = \mathrm{pol}(A_0)[\boldsymbol{a}^0]$ and $\pi^3 = \mathrm{pol}(A_3)[\boldsymbol{a}^3]$ of the vertices A_0 and A_3 (see Fig. 1), so

$$J = \pi^3 \cap A_0A_3, Q = \pi^3 \cap A_2A_3, E = \pi^3 \cap A_1A_3, \text{ and}$$
$$H = \pi^0 \cap A_0A_3, L = \pi^0 \cap A_0A_2, C = \pi^0 \cap A_0A_1$$

are proper points. Q lies on edge A_2A_3 so we can write $\mathbf{q} \sim c \cdot \mathbf{a}_3 + \mathbf{a}_2$ for some c real number. The corresponding Q point lies on $\boldsymbol{a}^3$ if and only if their scalar product is 0:

$$c \cdot \mathbf{a}_3\boldsymbol{a}^3 + \mathbf{a}_2\boldsymbol{a}^3 = 0 \Leftrightarrow c = -\frac{\mathbf{a}_2\boldsymbol{a}^3}{\mathbf{a}_3\boldsymbol{a}^3},$$

$$\mathbf{q} \sim -\frac{\mathbf{a}_2\boldsymbol{a}^3}{\mathbf{a}_3\boldsymbol{a}^3}\mathbf{a}_3 + \mathbf{a}_2 \sim \mathbf{a}_2(\mathbf{a}_3\boldsymbol{a}^3) - \mathbf{a}_3(\mathbf{a}_2\boldsymbol{a}^3) = \mathbf{a}_2h_{33} - \mathbf{a}_3h_{23}. \tag{3.3}$$

Similarly for the other vertices on π^3 and π^0 polar planes,

$$\begin{aligned}
\mathbf{j} &\sim \mathbf{a}_0(\mathbf{a}_3\boldsymbol{a}^3) - \mathbf{a}_3(\mathbf{a}_0\boldsymbol{a}^3) = \mathbf{a}_0h_{33} - \mathbf{a}_3h_{03},\\
\mathbf{e} &\sim \mathbf{a}_1(\mathbf{a}_3\boldsymbol{a}^3) - \mathbf{a}_3(\mathbf{a}_1\boldsymbol{a}^3) = \mathbf{a}_1h_{33} - \mathbf{a}_3h_{13},\\
\mathbf{h} &\sim \mathbf{a}_3(\mathbf{a}_0\boldsymbol{a}^0) - \mathbf{a}_0(\mathbf{a}_3\boldsymbol{a}^0) = \mathbf{a}_3h_0 - \mathbf{a}_0h_{03},\\
\mathbf{l} &\sim \mathbf{a}_2(\mathbf{a}_0\boldsymbol{a}^0) - \mathbf{a}_0(\mathbf{a}_2\boldsymbol{a}^0) = \mathbf{a}_2h_{00} - \mathbf{a}_0h_{02},\\
\mathbf{c} &\sim \mathbf{a}_1(\mathbf{a}_0\boldsymbol{a}^0) - \mathbf{a}_0(\mathbf{a}_1\boldsymbol{a}^0) = \mathbf{a}_1h_0 - \mathbf{a}_0h_{01}.
\end{aligned} \tag{3.4}$$

Set the orthoscheme in the model with coordinates

$$\begin{aligned}
&Q = (1,0,0,0), E = (1,0,y,0), J = (1,x,y,0),\\
&A_0 = (1,x,y,-z_0), A_1 = (1,0,y,-z_1), A_2 = (1,0,0,-z_2),\\
&H = (1,x,y,-z_H), L = (t_1x, t_1y, -t_1z_0 - (1-t_1)z_2),\\
&C = (t_2x, y, -t_2z_0 - (1-t_2)z_1),
\end{aligned}$$

for some $t_1, t_2 \in [0,1]$ (see Fig. 2), and using formulas (2.1) and (3.3-4):

$$\cosh(d(Q,E)) = \frac{-\langle \mathbf{q}, \mathbf{e}\rangle}{\sqrt{\langle \mathbf{q}, \mathbf{q}\rangle \langle \mathbf{e}, \mathbf{e}\rangle}} = \frac{h_{13}h_{23} - h_{12}h_{33}}{\sqrt{(h_{22}h_{33} - h_{23}^2)(h_{11}h_{33} - h_{13}^2)}},$$

$$\cosh(d(Q,J)) = \frac{-\langle \mathbf{q}, \mathbf{j}\rangle}{\sqrt{\langle \mathbf{q}, \mathbf{q}\rangle \langle \mathbf{j}, \mathbf{j}\rangle}} = \frac{h_{03}h_{23} - h_{02}h_{33}}{\sqrt{(h_{22}h_{33} - h_{23}^2)(h_0h_{33} - h_{03}^2)}},$$

$$\cosh(d(E,A_1)) = \frac{-\langle \mathbf{a}_1, \mathbf{e}\rangle}{\sqrt{\langle \mathbf{e}, \mathbf{e}\rangle \langle \mathbf{a}_1, \mathbf{a}_1\rangle}} = \sqrt{\frac{h_{11}h_{33} - h_{13}^2}{h_{11}h_{33}}},$$

$$\cosh(d(Q,A_2)) = \frac{-\langle \mathbf{a}_2, \mathbf{q}\rangle}{\sqrt{\langle \mathbf{q}, \mathbf{q}\rangle \langle \mathbf{a}_2, \mathbf{a}_2\rangle}} = \sqrt{\frac{h_{22}h_{33} - h_{23}^2}{h_{22}h_{33}}},$$

$$\cosh(d(J,H)) = \frac{-\langle \mathbf{j}, \mathbf{h}\rangle}{\sqrt{\langle \mathbf{j}, \mathbf{j}\rangle \langle \mathbf{h}, \mathbf{h}\rangle}} = \frac{-h_{02}h_{03}h_{33} + h_{03}^2h_{23}}{\sqrt{h_{00}h_{33}(h_{22}h_{33} - h_{23}^2)(h_{00}h_{33} - h_{03}^2)}}.$$

We know furthermore, that $0 = \langle \mathbf{h}, \mathbf{a}_0\rangle = \langle \mathbf{c}, \mathbf{a}_0\rangle = \langle \mathbf{l}, \mathbf{a}_0\rangle$, hence we can determine the coordinates of the vertices of a certain $\mathcal{F}_u^{(v,w)}$.

The images of this $\mathcal{F}_u^{(v,w)}$ doubly truncated orthoscheme under reflections on its facets provide the Coxeter tiling $\mathcal{T}_u^{(v,w)}$ with fundamental domain $\mathcal{F}_u^{(v,w)}$. We construct hyperball coverings as follows (see Fig. 4):

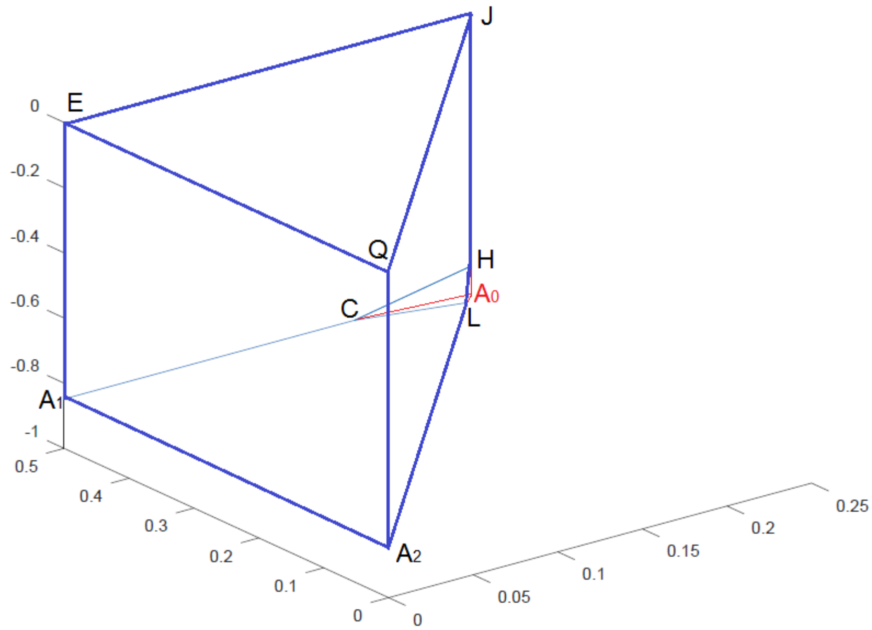


FIGURE 2. $\{7,3,7\}$ Coxeter orthoscheme: A_0 vertex truncated with HLC, A_3 vertex truncated with QEJ polar plane.

- Let QEJ the base hyperplane of a hyperball, and consider its piece bounded with QEJ plane, the planes perpendicular to this base hyperplane, and contain the edges QE, QJ, JE. Denote this hyperball-piece with $\mathcal{H}_1$, and its height parameter with h_1.
- Let HLC the base hyperplane of the other hyperball, and consider its piece bounded with HLC plane, the planes perpendicular to this base hyperplane, and contain the edges HL, HC, CL. Denote this hyperball-piece with $\mathcal{H}_2$, and its height parameter with h_2.

It is obvious, that these hyperballs cover the orthoscheme if and only if they cover all of its edges, that is what we will check in the different cases. So if the hyperballs cover the edges of $\mathcal{F}_u^{(v,w)}$, the images of $\mathcal{H}_1$ and $\mathcal{H}_1$ under the reflections on the orthoschemes facets will provide a $\mathcal{C}_u^{(v,w)}$ covering of $\mathbb{H}^3$.

Here we remark, that the density cannot be optimal if we use only one hyperball to cover the orthoscheme, i.e. $h_i = 0$ for the other one.

A necessary condition for optimality is that there is a certain point of one of the $QA_2, EA_1, JH, LA_2, CA_1, A_1A_2$ edges lying on surfaces of both hyperballs. Therefore, the optimal covering density will be realized if the above conditions stand. According to this, we distinguish 6 cases.

Definition 3.3. *The density of $\mathcal{C}_u^{(v,w)}$ covering:*

$$\delta(\mathcal{C}_u^{(v,w)}) = \frac{Vol(\mathcal{H}_1) + Vol(\mathcal{H}_2)}{Vol(\mathcal{F}_u^{(v,w)})}.$$

3.3. **To cover, or not to cover?** In order to decide whether a covering can be realized or not, we define the distance functions of the edges related to polar hyperplanes $\pi^0 = \text{pol}(A_0)[\boldsymbol{a}^0]$ and $\pi^3 = \text{pol}(A_3)[\boldsymbol{a}^3]$ of the orthoscheme. For example consider HLC plane, JH edge (see Fig. 2), and parametrize JH by the following way: let T a general point of JH, where $T(1, x, y, -tz_H),\ t \in [0, 1]$. Hence the distance function of JH from HLC at t, is the distance of T and H, so by (2.1):

$$d_{HLC}^{JH}(t) = \arccos\left(\frac{1 - x^2 - y^2 - tz_H^2}{\sqrt{(1 - x^2 - y^2 - z_H^2)(1 - x^2 - y^2 - t^2 z_H^2)}}\right).$$

Similarly we get functions $d_{HLC}^{LA_2}(t), d_{HLC}^{CA_1}(t), d_{HLC}^{QA_2}(t), d_{HLC}^{EA_1}(t), d_{HLC}^{A_1A_2}(t)$. In the other case consider QEJ plane, and JH edge for example. Here the function is given by the distance of $T(1, x, y, -tz_H),\ t \in [0, 1]$, and J:

$$d_{QEJ}^{JH}(t) = \arccos\left(\sqrt{\frac{1 - x^2 - y^2}{1 - x^2 - y^2 - (tz_H)^2}}\right).$$

And similarly for functions $d_{QEJ}^{QA_2}(t), d_{QEJ}^{EA_1}(t), d_{QEJ}^{LA_2}(t), d_{QEJ}^{CA_1}(t), d_{QEJ}^{A_1A_2}(t)$.

As we saw at the end of the previous section, we get optimal covering density, only if the surfaces of the two hyperballs intersect each other on an edge. So we choose an edge e (which is not on one of the polar planes), and one of its points $T(t)$ (parameterized with $t \in [0, 1]$), and say, that the T lies on the surface of both hyperballs. Hence we know the h_1, h_2 heights of the hyperballs, and we have to check, whether the distance of the points of the other edges from one of the hyperplanes are smaller than the corresponding h_i or not. We can determine the intersection points of a hyperball and an edge by using (2.2), and solving equations. If all of the points of an edge are closer to QEJ than h_1 or to HLC than h_2, than the covering is realized at T, and if this stands for all T in e, than the covering is realized at edge e ($e \in \{QA_2, EA_1, JH, LA_2, CA_1, A_1A_2\}$).

By the careful analysis of the above distance functions, using the help of a computer, we can say the following:

- The covering is realized at A_1A_2 edge (see Fig. 4).
- The covering is realized at QA_2 edge.
- The covering is realized at CA_1 edge.
- The covering isn't realized at EA_1 edge (the hyperballs don't cover QA_2 or A_1A_2 edge).
- The covering isn't realized at LA_2 edge (the hyperballs don't cover CA_1 or A_1A_2 edge).
- The covering isn't realized at HJ edge.

3.4. **Non-congruent coverings.** In this subsection, we consider the non-congruent coverings for the possible cases described in the previous subsection.

We can determine the areas of QEJ and HLC triangles, and if we settle $T(t)$ on the edge QA_2, then the heights of the hyperballs are

$$h_1(t) = d_{QEJ}^{QA_2}(t) \text{ and } h_2(t) = d_{HLC}^{QA_2}(t),$$

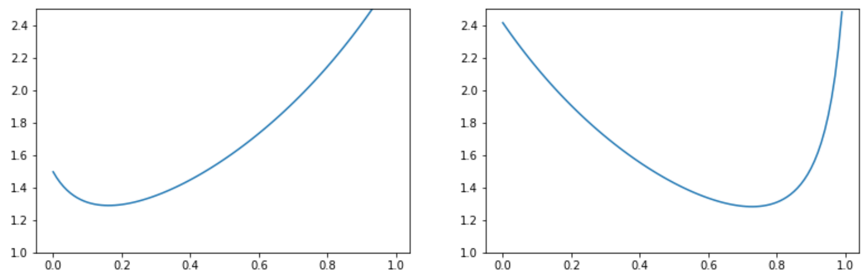
thus we can compute the density of the covering using (2.3), Theorem 3.2, and Definition 3.3. The covering density will be a function with variable t, and we can determine its minimum precisely by real analysis (see Fig. 3). The following table shows the minimal covering densities at QA_2 for different types of orthoschemes.

There are infinitely many types of doubly truncated orthoschemes, as we saw previously. Here we listed only some of the first elements of the four infinite sequences described in section 3. With the further increase of the parameters u, v, w, the density grows as we obtain it after careful analysis of the density function.

Type of orthoscheme	δ_{min}	h_1	h_2
$\mathcal{F}_3^{(7,3)}$	1.28943	0.92295	1.55521
$\mathcal{F}_3^{(8,3)}$	1.34248	0.67445	1.35737
$\mathcal{F}_4^{(5,4)}$	1.54311	0.73337	1.51710
$\mathcal{F}_4^{(6,4)}$	1.66605	0.52867	1.37017
$\mathcal{F}_5^{(4,5)}$	1.79576	0.77124	1.66724
$\mathcal{F}_5^{(5,4)}$	2.00292	0.42347	1.79770
$\mathcal{F}_6^{(4,5)}$	2.23585	0.53126	1.87500
$\mathcal{F}_6^{(5,4)}$	2.60090	0.31440	2.00574
$\mathcal{F}_7^{(3,7)}$	2.31671	1.08534	2.14790
$\mathcal{F}_7^{(4,5)}$	2.77700	0.42041	2.04284

The case of covering at CA_1 is the same, because of the symmetry of the orthoscheme. And see the optimal densities of the covering at the edge A_1A_2 in the following table.

Type of orthoscheme	δ_{min}	h_1	h_2
$\mathcal{F}_3^{(7,3)}$	1.38712	1.36405	1.36405
$\mathcal{F}_3^{(8,3)}$	1.45345	1.15039	1.15039
$\mathcal{F}_4^{(5,4)}$	1.36411	1.16974	1.16974
$\mathcal{F}_4^{(5,5)}$	1.41055	1.29237	0.85103
$\mathcal{F}_5^{(4,5)}$	1.31751	1.19095	1.19095
$\mathcal{F}_5^{(4,6)}$	1.34255	1.26048	0.95234
$\mathcal{F}_6^{(4,5)}$	1.34255	0.95234	1.26048
$\mathcal{F}_6^{(4,6)}$	1.35938	1.01481	1.01481
$\mathcal{F}_7^{(3,7)}$	1.26829	1.49903	1.49903
$\mathcal{F}_7^{(3,8)}$	1.28228	1.53709	1.22995



(A) $\delta(\mathcal{C}_3^{(7,3)})(t)$, when T lies on the QA_2 edge.

(B) $\delta(\mathcal{C}_7^{(3,8)})(t)$, when T lies on the A_1A_2 edge.

FIGURE 3

Notice that if $u = w$, than the optimal configuration belongs to congruent covering ($h_1 = h_2$).

3.4.1. *On non-extendable congruent and non-congruent hyperball coverings to parameters* $\{u, 3, 7\}$, $(6 < u < 7, u \in \mathbb{R})$. We can investigate the coverings related to orthoschemes $\mathcal{F}_u^{(3,7)}$, $(6 < u < 7, u \in \mathbb{R})$, and here we consider the coverings in general, i.e. both congruent and non-congruent cases.

In this case, the images of the orthoschemes under reflections on their facets do not fill the hyperbolic space, but it provides a thinner, local covering, which can not be extended to the entire $\mathbb{H}^3$. Here the previous computations are the same, and the density function is two-variable, whose minimum can be determined numerically after accurate analysis.

Theorem 3.4. *The non-congruent hypersphere coverings in* $\mathcal{C}_u^{(3,7)}$, $(6 < u < 7, u \in \mathbb{R})$ *attain their minimum density at* $u \approx 6.45953$ *with density* ≈ 1.26454 *where the heights of hyperspheres are* $h_1 \approx 1.50377$, *and* $h_2 \approx 1.26423$ *and their common point lies on* A_1A_2 *edge.*

Remark:

(1) Notice, that the parameter u, where the minimum is attained ≈ 6.45953, is very close to the corresponding parameter in [8], where it was ≈ 6.45962.
(2) The above covering density is smaller than the ≈ 1.280 density belonging to Fejes Tóth László, Böröczky Károly, but this hyperball covering can not be extended to the entire space.

3.5. **Congruent coverings.** In this subsection, we consider the congruent coverings for the possible cases described in Subsection 3.3.

In this case, we are looking for the T point on A_1A_2, CA_1, QA_2 edges, which is equal distance from HLC and QEJ. Investigating the above distance function, this point doesn't exists on CA_1 and QA_2, and on the third edge, we can find it by solving the equation $d^{A_1A_2}_{HLC}(t) = d^{A_1A_2}_{QEJ}(t)$, see Fig. 4 for visualization. In view of T, we can determine the data of the covering as above. Here we listed just finitely many types of orthoschemes as well, but as above, the further types can't provide smaller densities as we obtain it after careful analysis of the density function.

Type of orthoscheme	δ_{min}	h_1	h_2
$\mathcal{F}_3^{(7,3)}$	1.38712	1.36405	1.36405
$\mathcal{F}_3^{(8,3)}$	1.45345	1.15039	1.15039
$\mathcal{F}_4^{(5,4)}$	1.36411	1.16974	1.16974
$\mathcal{F}_4^{(6,4)}$	1.45714	0.99583	0.99583
$\mathcal{F}_5^{(4,5)}$	1.31751	1.19095	1.19095
$\mathcal{F}_5^{(4,6)}$	1.45345	1.13375	1.13375
$\mathcal{F}_6^{(4,5)}$	1.45345	1.13375	1.13375
$\mathcal{F}_6^{(4,6)}$	1.35938	1.01481	1.01481
$\mathcal{F}_7^{(3,7)}$	1.26829	1.49903	1.49903
$\mathcal{F}_7^{(3,8)}$	1.36586	1.39916	1.39916

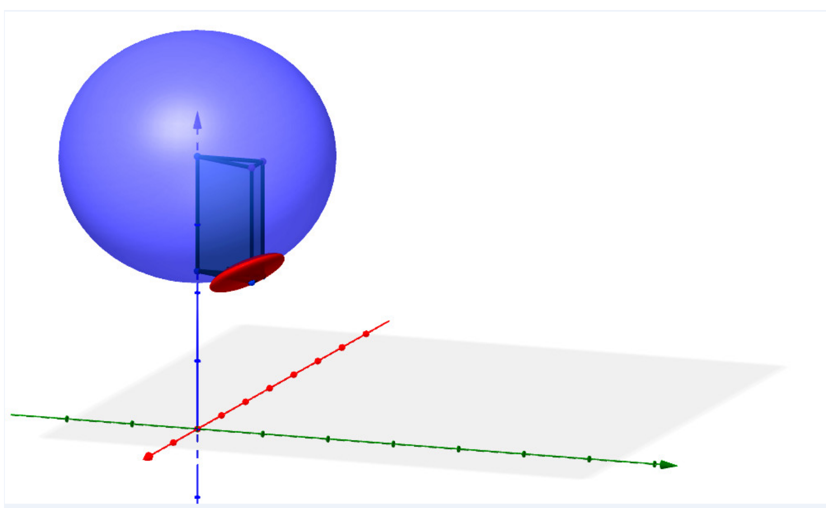


FIGURE 4. Visualization of $\mathcal{C}_7^{(3,7)}$ with density ≈ 1.26829

We summarized our results in the following theorem.

Theorem 3.6. *In the hyperbolic space* $\mathbb{H}^3$*, between the congruent and non-congruent hyperball coverings generated by doubly truncated Coxeter orthoschemes, the* $\mathcal{C}_7^{(3,7)}$ *congruent hyperball configuration provides the thinnest covering with density* ≈ 1.26829*, which is the so far known smallest ball covering density in* $\mathbb{H}^3$*.*

4. Hypercycle covering in hyperbolic plane

In the hyperbolic plane ($n = 2$) I. Vermes proved, that the lower bound for congruent hypercycle covering density is $\sqrt{12}/\pi$ [36]. However, there are no results related to the thinnest hypercycle coverings with non-congruent hypercycles. The investigation of the density of non-congruent hypersphere coverings is generally not easy.

Here we will prove, using the results of our paper [8], that the theoretic lower bound $\sqrt{12}/\pi$ for congruent hypercycle coverings can be arbitrary approximated with non-congruent hypercycle coverings related to doubly truncated orthoschemes.

Theorem 4.1. *Let $A(1, 0, a)$ and $B(1, b, 0)$ outer points related to the Beltrami-Cayley-Klein circle model (see Fig. 5), and the AB line intersects the model circle. Let the base lines of two hypercycles be OE and FC, and the corresponding hypercycles through the midpoint J of segment CD generate a covering configuration $\mathcal{C}_{a,b}\,(1/2)$ in truncated orthosceme $FCDEO$. Then*

$$\lim_{(a,b)\to(1,\infty)} \delta\left(\mathcal{C}_{a,b}\left(\frac{1}{2}\right)\right) = \frac{\sqrt{12}}{\pi}.$$

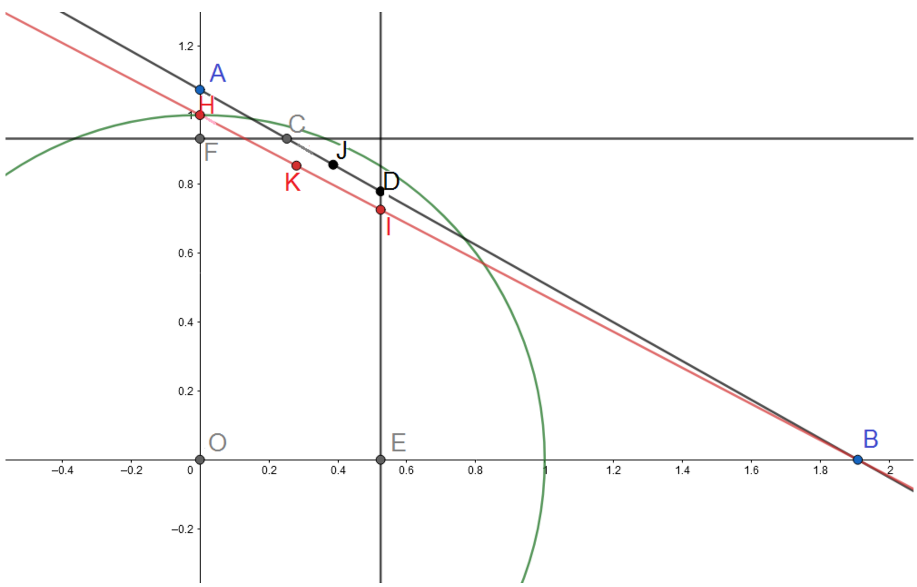


Figure 5.
$FCDEO$ doubly truncated Coxeter orthoscheme

Proof. Here let the base hyperlines of the two hypercycles OE and FC, and both hypercycles pass through the point J that is the midpoint of CD. This yields obviously a covering, denote it with $\mathcal{C}_{a,b}\,(1/2)$. If $a \to 1$ then $A \to H$, $CD \to HI$, and the K point on HI arises as the limit of the J

midpoints of CD. Hence the orthoscheme tends to $HIEO$ that is a simple asymptotic, simple truncated orthoscheme. The hypercycle, whose base line is CF, tends to a horocycle centered at H. The hypercycle is produced as the images of a point under reflections on lines perpendicular to a certain line CF, i.e. lines passing through the polar point (A) of a certain line CF. The horocycle is produced as images of a point under reflections on parallel lines, i.e. lines passing through a point at infinity (H). It means, that if A tends to H, then the hypercycle tends to a horocycle.

So if $a \to 1$, we get a hyp-hor covering, investigated in [8]. We recall Theorem 3.2 of [8] using the denotation of the present paper:

Theorem 4.2 (Eper and Szirmai [8]). *Let $B(1, b, 0)$ outer point of the simple truncated orthoscheme, and $\mathcal{C}_b\,(1/2)$ denote the hyp-hor covering of simple truncated orthoscheme $HIEO$, with cycles passing through the point K. Then*

$$\lim_{b\to\infty} \delta\left(\mathcal{C}_b^1\left(\frac{1}{2}\right)\right) = \frac{\sqrt{12}}{\pi}.$$

and $\delta\left(\mathcal{C}_b^1\,(1/2)\right) > \sqrt{12}/\pi$ for parameter $b > 1$.

And now according to the results of this theorem, we obtain the following:

$$\begin{aligned}
&\lim_{(a,b)\to(1,\infty)} \delta\left(\mathcal{C}_{a,b}\left(\frac{1}{2}\right)\right)\\
&= \lim_{b\to\infty}\lim_{a\to 1} \delta\left(\mathcal{C}_{a,b}\left(\frac{1}{2}\right)\right)\\
&= \lim_{b\to\infty} \delta\left(\mathcal{C}_b^1\left(\frac{1}{2}\right)\right) = \frac{\sqrt{12}}{\pi},
\end{aligned}$$

that means that density $\sqrt{12}/\pi$ can be arbitrarily approximated with hypercycle coverings related to doubly truncated orthoschemes. □

Remark: The general investigation of planar non-congruent hypersphere packings and coverings related to the doubly truncated orthoschemes is discussed in a forthcoming paper.

References

1. K. Bezdek, *Sphere packings revisited*, Eur. J. Combin., **27/6** (2006), 864–883.
2. J. Bolyai, *The science of absolute space*, Austin, Texas (1891).
3. J. Böhm and E. Hertel, *Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung*, Birkhäuser, Basel (1981).
4. K. Böröczky, *Gömbelhelyezések állandó görbületü terekben I.*, Math. Lapok, **25**, (1974), 265–306.
5. ______, *Gömbelhelyezések állandó görbületü terekben II.*, Math. Lapok, **26**, (1975), 67–90.
6. ______, *Packing of spheres in spaces of constant curvature*, Acta Math. Acad. Sci. Hungar., **32** (1978), 243–261.
7. K. Böröczky and A. Florian, *Über die dichteste Kugelpackung im hyperbolischen Raum*, Acta Math. Acad. Sci. Hungar., **15** (1964), 237–245.

8. M. Eper and J. Szirmai, *Coverings with horo- and hyperballs generated by simply truncated orthoschemes*, Novi Sad J. Math., to appear (2021).
9. L. F. Tóth, *Regular Figures*, Macmillian (New York), 1964.
10. H.-C., Im Hof, *A class of hyperbolic Coxeter groups*, Expo. Math., **3** (1985), 179–186.
11. ———, *Napier cycles and hyperbolic Coxeter groups*, Bull. Soc. Math. Belgique, **42** (1990), 523–545.
12. J. Kepler, *Strena seu de nive sexangula*, Frankfurt, Germany: Tampach (1611) Reprinted in Gesammelte Werke (Ed. M. Caspar and F. Hammer, **4** Oxford, England: Clarendon Press (1966)
13. R. Kellerhals, *The dilagorithm and volumes of hyperbolic polytopes,* AMS Mathematical Surveys and Monographs, **37** (1991), 301–336.
14. ———, *On the volume of hyperbolic polyhedra,* Math. Ann., **245** (1989), 541–569.
15. R. T. Kozma and J. Szirmai, *Optimally dense packings for fully asymptotic Coxeter tilings by horoballs of different types*, Monatsh. Math., **168/1** (2012), 27–47.
16. ———, *New lower bound for the optimal ball packing density of hyperbolic 4-space*, Discrete Comput. Geom., **53** (2015), 182–198, DOI: 10.1007/s00454-014-9634-1.
17. ———, *New horoball packing density lower bound in hyperbolic 5-space*, Geometriae Dedicata, **206/1** (2020), 1–25, DOI: 10.1007/s10711-019-00473-x.
18. ———, *Horoball packing density lower bounds in higher dimensional hyperbolic n-space for* $6 \leq n \leq 9$, Submitted manuscript, (2020), arXiv:1907.00595.
19. ———, *The structure and visualization of optimal horoball packings in 3-dimensional hyperbolic space*, Submitted manuscript, (2020), arXiv:1601.03620, (Appendix: http://homepages.math.uic.edu/rkozma/SVOHP.html)).
20. E. Molnár, M. Stojanovic and J. Szirmai, *Non-fundamental trunc-simplex tilings and their optimal hyperball packings and coverings in hyperbolic space*, Submitted manuscript, (2020).
21. E. Molnár and J. Szirmai, *Top dense hyperbolic ball packings and coverings for complete Coxeter orthoscheme groups*, Publications de l'Institut Mathématique, **103(117)** (2018), 129–146, DOI: 10.2298/PIM1817129M.
22. J. Szirmai, *Congruent and non-congruent hyperball packings related to doubly truncated Coxeter orthoschemes in hyperbolic 3-space,* Acta Univ. Sapientiae, Mathematica, **11**, 2 (2019), 437–459.
23. ———, *Horoball packings to the totally assymptotic regular simplex in the hyperbolic n-space*, Aequat. Math., **85**, (2013), 471–482, DOI: 10.1007/s00010.012-0158-6.
24. ———, *Horoball packings and their densities by generalized simplicial density function in the hyperbolic space*, Acta Math. Hung., **136/1-2**, (2012), 39–55, DOI: 10.1007/s10474-012-0205-8.
25. ———, *Horoball packings related to the 4-dimensional hyperbolic 24 cell honeycomb* $\{3, 4, 3, 4\}$, Filomat, **32/1**, (2018), 87–100, DOI: 10.2298/FIL1801087S, arXiv:1502.02107.
26. ———, *Hyperball packings related to cube and octahedron tilings in hyperbolic space*, Contributions to Discrete Mathematics, **15/2**, (2020), 42–59.
27. ———, *The p-gonal prism tilings and their optimal hypersphere packings in the hyperbolic 3-space*, Acta Math. Hungar. **111 (1-2)** (2006), 65–76.
28. ———, *The regular prism tilings and their optimal hyperball packings in the hyperbolic n-space*, Publ. Math. Debrecen **69 (1-2)** (2006), 195–207.

29. ______, *The least dense hyperball covering to the regular prism tilings in the hyperbolic n-space*, Ann. Mat. Pur. Appl., **195** (2016), 235–248, DOI: 10.1007/s10231-014-0460-0.
30. ______, *Hyperball packings in hyperbolic* 3*-space*, Mat. Vesnik **70/3** (2018), 211–221.
31. ______, *Packings with horo- and hyperballs generated by simple frustum orthoschemes*, Acta Math. Hungar., **152/2** (2017), 365–382, DOI:10.1007/s10474-017-0728-0.
32. ______, *Density upper bound of congruent and non-congruent hyperball packings generated by truncated regular simplex tilings*, Rend. Circ. Mat. Palermo (2), **67** (2018), 307–322, DOI: 10.1007/s12215-017-0316-8.
33. ______, *Decomposition method related to saturated hyperball packings*, Ars Math. Contemp., **16** (2019), 349–358, DOI: 10.26493/1855-3974.1485.0b1.
34. ______, *Upper bound of density for packing of congruent hyperballs in hyperbolic* $3-$*space*, Submitted manuscript, (2020), arXiv:1812.06785.
35. I. Vermes, *Ausfüllungen der hyperbolischen Ebene durch kongruente Hyperzykelbereiche*, Period. Math. Hungar. (1979) **10/4**, 217–229.
36. ______, *Über reguläre Überdeckungen der Bolyai-Lobatschewskischen Ebene durch kongruente Hyperzykelbereiche*, Periodica Polytechnica. Engineering. Mechanical Engineering, **25/3** (1981), 249–261.
37. ______, *Bemerkungen zum Problem der dünnesten Überdeckungen der hyperbolischen Ebene durch kongruente Hyperzykelbereiche*, Studia Sci. Math. Hungar., **23**, (1988), 1–6.

Budapest University of Technology and Economics Institute of Mathematics, Department of Geometry, H-1521, Hungary
E-mail address: epermiklos@gmail.com

Budapest University of Technology and Economics Institute of Mathematics, Department of Geometry, H-1521, Hungary
E-mail address: szirmai@math.bme.hu