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THE SIMPLICITY INDEX OF TOURNAMENTS

ABDERRAHIM BOUSSAÏRI, SOUFIANE LAKHLIFI, AND IMANE TALBAOUI

Abstract. An n-tournament T with vertex set V is simple if there is no subset M of V such that $2 \leq |M| \leq n-1$ and for every $x \in V \setminus M$, either $M \to x$ or $x \to M$. The simplicity index of an n-tournament T is the minimum number $s(T)$ of arcs whose reversal yields a nonsimple tournament. Müller and Pelant (1974) proved that $s(T) \leq (n-1)/2$, and that equality holds if and only if T is doubly regular. As doubly regular tournaments exist only if $n \equiv 3 \pmod 4$, $s(T) < (n-1)/2$ for $n \not\equiv 3 \pmod 4$. In this paper, we study the class of n-tournaments with maximal simplicity index for $n \not\equiv 3 \pmod 4$.

1. Introduction

A *tournament* T consists of a finite set V of *vertices* together with a set A of ordered pairs of distinct vertices, called *arcs*, such that for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. Such a tournament is denoted by $T = (V, A)$. Given $x \neq y \in V$, we say that x *dominates* y and we write $x \to y$ when $(x, y) \in A$. Similarly, given two disjoint subsets X and Y of V, we write $X \to Y$ if $x \to y$ holds for every $(x, y) \in X \times Y$. Throughout this paper, we mean by an *n-tournament* a tournament with n vertices.

A tournament is *regular* if there is an integer $k \geq 1$ such that each vertex dominates exactly k vertices. It is *doubly regular* if there is an integer $k \geq 1$ such that every unordered pair of vertices dominates exactly k vertices.

A tournament is *transitive*, if for any vertices x, y and z, $x \to y$ and $y \to z$ implies that $x \to z$. A tournament $T = (V, A)$ is *reducible* if V admits a bipartition $\{X, Y\}$ such that $X \to Y$. The notion of simple tournament was introduced by Fried and Lakser [8], it was motivated by questions in algebra. It is closely related to modular decomposition [9] which involves the notion of module. Recall that a *module* of a tournament $T = (V, A)$ is a subset M of V such that for every $x \in V \setminus M$ either $M \to \{x\}$ or $\{x\} \to M$. For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are modules of T called *trivial*

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Corresponding author: Abderrahim Boussaïri.

modules. An n-tournament is *simple* [6, 15] (or prime [4] or primitive [5] or indecomposable [10, 17]) if $n \geq 3$ and all its modules are trivial. The simple tournaments with at most 5 vertices are shown in Figure 1. A tournament is *decomposable* if it admits a nontrivial module.

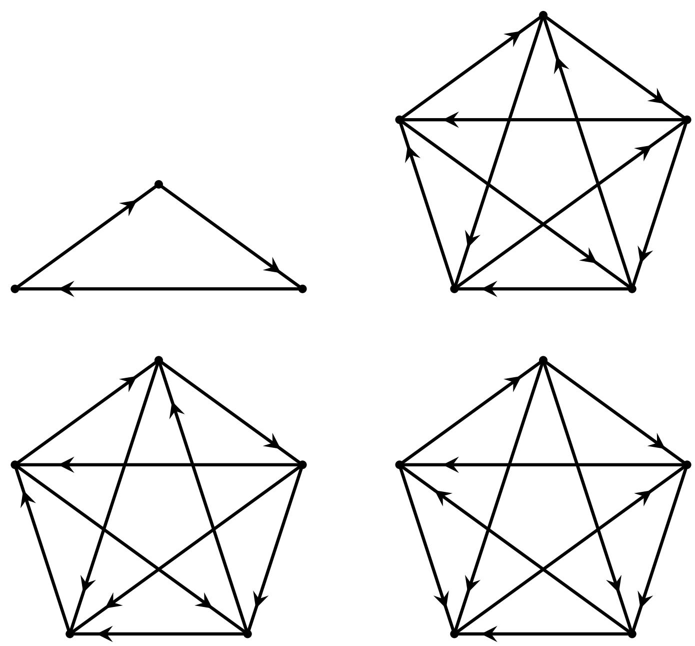

FIGURE 1. The simple tournaments with at most 5 vertices

Given an n-tournament T, the *Slater index* $i(T)$ of T is the minimum number of arcs that must be reversed to make T transitive [18]. It is not difficult to see that $i(T) \leq n(n-1)/4$. However, we do not know an exact determination of the upper bound of $i(T)$. Erdős and Moon [7] proved that this bound is asymptotically equal to $n^2/4$. Recently, Satake [16] proved that the Slater index of doubly regular n-tournaments is at least

$$\frac{n(n-1)}{4} - n^{\frac{3}{2}} \log_2(2n).$$

Kirkland [11] defined the *reversal index* $i_R(T)$ of a tournament T as the minimum number of arcs whose reversal makes T reducible. Clearly, $i_R(T) \leq i(T)$. Kirkland [11] proved that $i_R(T) \leq \lfloor (n-1)/2 \rfloor$ and characterized all the tournaments for which equality holds.

The indices above can be interpreted in terms of distance between tournaments. The *distance* $d(T_1, T_2)$ between two tournaments T_1 and T_2 with the same vertex set is the number of pairs $\{x, y\}$ of vertices for which the arc between x and y has not the same direction in T_1 and T_2. Let $\mathcal{F}$ be a family of tournaments with vertex set V. The distance from a tournament T to the family $\mathcal{F}$ is $d(T, \mathcal{F}) = \min\{d(T, T') : T' \in \mathcal{F}\}$. If $\mathcal{F}$ is the family of transitive tournaments on V, then $i(T) = d(T, \mathcal{F})$. If $\mathcal{F}$ is the family of reducible tournaments on V, then $i_R(T) = d(T, \mathcal{F})$.

By considering the family of decomposable tournaments, we obtain the simplicity index introduced by Müller and Pelant [15]. Precisely, consider an n-tournament T, where $n \geq 3$. The *simplicity index* $s(T)$ of T (also called the *arrow-simplicity* of T in [15]) is the minimum number of arcs that must be reversed to make T nonsimple. For example, the tournaments shown in Figure 1 have simplicity index 1. Obviously, $s(T) \leq i_R(T)$ and $s(T) \leq (n-1)/2$. Müller and Pelant proved that $s(T) = (n-1)/2$ if and only if T is doubly regular.

A dual notion of the simplicity index is the decomposability index [2], which is obtained by considering the family of simple tournaments.

In this paper, we provide an upper bound for $s(T)$, where T is an n-tournament for $n \not\equiv 3 \pmod 4$. More precisely, we obtain the following result.

Theorem 1.1. *Given an n-tournament T, the following statements hold*

(1) *if $n = 4k+2$, then $s(T) \leq 2k$;*
(2) *if $n = 4k+1$, then $s(T) \leq 2k-1$;*
(3) *if $n = 4k$, then $s(T) \leq 2k-2$.*

To prove that the bounds in this theorem are the best possible, we use the double regularity as follows.

Theorem 1.2. *Let $l \in \{1, 2, 3\}$. Consider a doubly regular tournament T of order $4k+3$, where $k \geq l$. The simplicity index of a tournament obtained from T by removing l vertices is $(2k+1) - l$.*

As shown by the next result, the opposite direction in Theorem 1.2 holds when $l = 1$.

Theorem 1.3. *Given a tournament T with $4k+2$ vertices, where $k \geq 1$, if $s(T) = 2k$, then T is obtained from a doubly regular tournament by removing one vertex.*

The existence of doubly regular tournaments is equivalent to the existence of skew-Hadamard matrices [3]. Wallis [20] conjectured that $n \times n$ skew-Hadamard matrices exist if and only if $n = 2$ or n is divisible by 4. Infinite families of skew-Hadamard matrices can be found in [12].

The most known examples of a doubly regular tournament are obtained from Paley construction. For a prime power $q \equiv 3 \pmod 4$, the *Paley tournament* of order q is the tournament whose vertex set is the finite field $\mathbb{F}_q$, such that x dominates y if and only if $x - y$ is a nonzero quadratic residue in $\mathbb{F}_q$.

2. Preliminaries

Let $T = (V, A)$ be an n-tournament and let $x \in V$. The *out-neighborhood* of x is

$$N_T^+(x) := \{y \in V : x \to y\},$$

and the *in-neighborhood* of x is

$$N_T^-(x) := \{y \in V : y \to x\}.$$

The *out-degree* of x (resp. the *in-degree* of x) is

$$\delta_T^+(x) := \left|N_T^+(x)\right| \text{ (resp. } \delta_T^-(x) := \left|N_T^-(x)\right|\text{)}.$$

The *out-degree* of x is also called the *score* of x in T. Recall that

$$\sum_{z \in V} \delta_T^+(z) = \sum_{z \in V} \delta_T^-(z) = \frac{n(n-1)}{2}. \tag{2.1}$$

A tournament is *near-regular* if there exists an integer $k > 0$ such that the out-degree of every vertex equals k or $k-1$.

Remark: Let T be an n-tournament. It follows from (2.1) that

(1) T is regular if and only if n is odd and every vertex has out-degree $(n-1)/2$;
(2) T is near-regular if and only if n is even and T has $n/2$ vertices of out-degree $n/2$ and $n/2$ vertices of out-degree $(n-2)/2$.

Notation. *Let $T = (V, A)$ be a near-regular tournament of order $4k+2$. We can partition V into two $(2k+1)$-subsets,*

$$V_{\text{even}} := \{z \in V, \delta_T^+(z) = 2k\} \text{ and } V_{\text{odd}} := \{z \in V, \delta_T^+(z) = 2k+1\}.$$

Let x, y be distinct vertices of an n-tournament $T = (V, A)$. The set $V \setminus \{x, y\}$ can be partitioned into four subsets:

$$N_T^+(x) \cap N_T^+(y), \qquad N_T^-(x) \cap N_T^-(y),$$

$$N_T^+(x) \cap N_T^-(y), \qquad N_T^-(x) \cap N_T^+(y).$$

The *out-degree* (resp. the *in-degree*) of (x, y) is

$$\delta_T^+(x, y) := \left|N_T^+(x) \cap N_T^+(y)\right| \text{ (resp. } \delta_T^-(x, y) := \left|N_T^-(x) \cap N_T^-(y)\right|\text{)}.$$

The elements of $(N_T^+(x) \cap N_T^-(y)) \cup (N_T^-(x) \cap N_T^+(y))$ are called *separators* of x, y and their number is denoted by $\sigma_T(x, y)$.

Lemma 2.3. *Let T be an n-tournament with vertex set V. For any $x \neq y \in V$, we have*

- $\sigma_T(x, y) + \delta_T^-(x, y) + \delta_T^+(x, y) = n - 2$;
- $\delta_T^-(x, y) - \delta_T^+(x, y) = \delta_T^-(x) - \delta_T^+(y)$.

In particular, if T is regular, then for any $x \neq y \in V$, $\delta_T^-(x, y) = \delta_T^+(x, y)$.

Proof. The first assertion is obvious. For the second assertion, we have

$$\left|N_T^-(x)\right| = \left|N_T^-(x) \cap N_T^-(y)\right| + \left|N_T^-(x) \cap N_T^+(y)\right| + \left|N_T^-(x) \cap \{y\}\right|$$

and

$$\left|N_T^+(y)\right| = \left|N_T^+(y) \cap N_T^+(x)\right| + \left|N_T^+(y) \cap N_T^-(x)\right| + \left|N_T^+(y) \cap \{x\}\right|.$$

Moreover, $y \in N_T^-(x)$ if and only if $x \in N_T^+(y)$. Then

$$\left|N_T^-(x) \cap \{y\}\right| = \left|N_T^+(y) \cap \{x\}\right|$$

and hence

$$\left|N_T^-(x) \cap N_T^-(y)\right| - \left|N_T^+(x) \cap N_T^+(y)\right| = \left|N_T^-(x)\right| - \left|N_T^+(y)\right|. \qquad \square$$

Let $T = (V, A)$ be a tournament. For each vertex $z \in V$, we have

$$\delta_T^-(z)\delta_T^+(z) = \left|\left\{\{x, y\} \in \binom{V}{2} : z \in (N_T^-(x) \cap N_T^+(y)) \cup (N_T^+(x) \cap N_T^-(y))\right\}\right|.$$

By double-counting, we obtain

$$\sum_{z \in V} \delta_T^+(z)\delta_T^-(z) = \sum_{\{x,y\} \in \binom{V}{2}} \sigma_T(x, y). \tag{2.2}$$

In the next proposition, we give some basic properties of doubly regular tournaments. For the proof, see [15].

Proposition 2.4. *Let $T = (V, A)$ be a doubly regular n-tournament. There exists $k \geq 0$ such that $n = 4k + 3$, T is regular, and for all $x, y \in V$ such that $x \to y$, we have*

$$\left|N_T^+(x) \cap N_T^+(y)\right| = \left|N_T^-(x) \cap N_T^-(y)\right| = \left|N_T^+(x) \cap N_T^-(y)\right| = k$$

$$\textit{and} \quad \left|N_T^-(x) \cap N_T^+(y)\right| = k + 1.$$

3. Proof of Theorem 1.1

Let $T = (V, A)$ be a tournament. Given a subset B of A, we denote by $\mathrm{Inv}(T, B)$ the tournament obtained from T by reversing all the arcs of B. We also use the following notation:

$$\delta_T^+ = \min\left\{\delta_T^+(x) : x \in V\right\}, \qquad \delta_T^- = \min\left\{\delta_T^-(x) : x \in V\right\},$$

$$\delta_T = \min(\delta_T^+, \delta_T^-), \qquad \sigma_T = \min\{\sigma_T(x, y) : x \neq y \in V\}.$$

The next proposition provides an upper bound of the simplicity index of a tournament.

Proposition 3.1. *For a tournament $T = (V, A)$ with at least 3 vertices, we have $s(T) \leq \min(\delta_T, \sigma_T)$.*

Proof. Let $x \in V$. Clearly, the subset $V \setminus \{x\}$ is a nontrivial module of $\mathrm{Inv}(T, \{x\} \times N_T^+(x))$ and $\mathrm{Inv}(T, N_T^-(x) \times \{x\})$. It follows that

$$s(T) \leq \min_{x \in V}(\delta_T^+(x), \delta_T^-(x)) = \delta_T.$$

Now, consider an unordered pair $\{x, y\}$ of vertices of T and let

$$B := \left(\{x\} \times \left((N_T^+(x) \cap N_T^-(y)\right) \cup \left(N_T^+(y) \cap N_T^-(x)\right) \times \{x\}\right).$$

Clearly, $\{x, y\}$ is a module of $\mathrm{Inv}(T, B)$. It follows that

$$s(T) \leq |B| = \left|N_T^+(x) \cap N_T^-(y)\right| + \left|N_T^+(y) \cap N_T^-(x)\right| = \sigma_T(x, y).$$

Hence, $s(T) \leq \sigma_T$. □

In addition to the previous proposition, the proof of Theorem 1.1 requires the following lemma.

Lemma 3.2. *Given an n-tournament $T = (V, A)$ with $n \geq 2$, we have*

$$\delta_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } \sigma_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. For every $x \in V$, we have $\min\left(\delta_T^+(x), \delta_T^-(x)\right) \leq (n-1)/2$. Thus,

$$\delta_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Now, to verify that $\sigma_T \leq \lfloor (n-1)/2 \rfloor$, observe that

$$\sigma_T \leq \frac{1}{\binom{|V|}{2}} \sum_{\{x,y\} \in \binom{V}{2}} \sigma_T(x, y).$$

It follows from (2.2) that

$$\begin{aligned}
\sigma_T &\leq \frac{2}{n(n-1)} \sum_{z \in V} \delta_T^+(z)\delta_T^-(z) \\
&\leq \frac{2}{n(n-1)} \sum_{z \in V} \left(\frac{\delta_T^+(z) + \delta_T^-(z)}{2}\right)^2 \\
&\leq \frac{(n-1)}{2}.
\end{aligned}$$

□

Proof of Theorem 1.1. For the first statement, suppose that $n = 4k+2$. By Proposition 3.1 and Lemma 3.2, we have

$$s(T) \leq \delta_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor = 2k.$$

For the second statement, suppose that $n = 4k+1$. By Proposition 3.1, $s(T) \leq \delta_T$. If T is not regular, then $\delta_T < (n-1)/2$ and hence $s(T) \leq 2k-1$. Suppose that T is regular and let $x \neq y \in V$. By Lemma 2.3,

$$\sigma_T(x, y) = n - 2 - \delta_T^-(x, y) - \delta_T^+(x, y) \text{ and } \delta_T^-(x, y) = \delta_T^+(x, y).$$

Therefore, $\sigma_T(x,y)$ is odd, and hence σ_T is odd as well. By Lemma 3.2, $\sigma_T \leq \lfloor (n-1)/2 \rfloor = 2k$. Since σ_T is odd, we obtain $\sigma_T \leq 2k-1$. It follows from Proposition 3.1 that $s(T) \leq 2k-1$.

For the third statement, suppose that $n = 4k$. If T is not near-regular, then $\delta_T < 2k-1$, and hence $s(T) \leq 2k-2$ by Proposition 3.1. Suppose that T is near-regular. By Remark 2.1, for every $z \in V$, $\delta_T^+(z) \in \{2k, 2k-1\}$. It follows from (2.2) that

$$\sum_{\{x,y\}\in\binom{V}{2}} \sigma_T(x,y) = \sum_{z\in V} \delta_T^+(z)\delta_T^-(z) = 8k^2(2k-1). \tag{3.1}$$

Thus, we obtain

$$\begin{aligned}
\sigma_T &\leq \frac{1}{\binom{|V|}{2}} \sum_{\{x,y\}\in\binom{V}{2}} \sigma_T(x,y)\\
&\leq \frac{2}{4k(4k-1)} 8k^2(2k-1)\\
&\leq (2k-1) + \frac{2k-1}{4k-1}\\
&\leq 2k-1.
\end{aligned}$$

Since $s(T) \leq \sigma_T$ by Proposition 3.1, we obtain $s(T) \leq \sigma_T \leq 2k-1$. Seeking a contradiction, suppose that $s(T) = 2k-1$. We obtain $\sigma_T = 2k-1$. Let $x \in V_{\text{even}}$ and $y \in V_{\text{odd}}$ (see Notation 2.2). It follows from Lemma 2.3 that $\sigma_T(x,y)$ is even and hence $\sigma_T(x,y) \geq 2k$. Thus, there are at least $(2k)^2$ unordered pairs $\{x,y\}$ satisfying $\sigma_T(x,y) \geq 2k$. For the other $2\binom{2k}{2}$ unordered pairs, we have $\sigma_T(x,y) \geq \sigma_T = 2k-1$. It follows that

$$\sum_{\{x,y\}\in\binom{V}{2}} \sigma_T(x,y) \geq 2\binom{2k}{2}(2k-1) + (2k)^2(2k) > 8k^2(2k-1),$$

which contradicts (3.1). Consequently, $s(T) \leq 2k-2$. □

4. Proof of Theorem 1.2

To begin, recall that a *graph* is defined by a vertex set V and an edge set E. Two distinct vertices x and y of G are *adjacent* if $\{x,y\} \in E$. For a vertex x in G, the set

$$N_G(x) := \{y \in V : \{x,y\} \in E\}$$

is called the *neighborhood* of x in G. The *degree* of x is $\delta_G(x) := |N_G(x)|$.

Let $T = (V,A)$ be a tournament. To each subset C of V, we associate a graph in the following way. Denote by $s_C(T)$ the minimum number of arcs that must be reversed to make C a module of T. Clearly,

$$s(T) = \min\{s_C(T) : 2 \leq |C| \leq n-1\}. \tag{4.1}$$

A graph $G = (V, E)$ is called a *decomposability graph* for C if $|E| = s_C(T)$ and C is a module of the tournament

$$\mathrm{Inv}(T, \{(x, y) \in A : \{x, y\} \in E\})$$

obtained from T by reversing the arc between x and y for each edge $\{x, y\}$ of G. In the next lemma, we provide some of the properties of decomposability graphs.

Lemma 4.1. *Let $T = (V, A)$ be a n-tournament and let C be a subset of V such that $2 \leq |C| \leq n - 1$. Given a decomposability graph $G = (V, E)$ for C, the following assertions hold*

- *G is bipartite with bipartition $\{C, V \setminus C\}$;*
- *for each $x \in V \setminus C$, $N_G(x) = N_T^+(x) \cap C$ or $N_G(x) = N_T^-(x) \cap C$, and $\delta_G(x) = \min\left(\left|N_T^-(x) \cap C\right|, \left|N_T^+(x) \cap C\right|\right)$.*

Proof. The first assertion follows from the minimality of $|E| = s_C(T)$. For the second assertion, consider $x \in V \setminus C$. Since C is a module of the tournament $\mathrm{Inv}(T, \{(x, y) \in A : \{x, y\} \in E\})$, we have

$$N_G(x) = N_T^+(x) \cap C \text{ or } N_G(x) = N_T^-(x) \cap C.$$

Furthermore, it follows from the minimality of $|E| = s_C(T)$ that

$$\delta_G(x) = \min\left(\left|N_T^-(x) \cap C\right|, \left|N_T^+(x) \cap C\right|\right).$$

□

The next proposition is useful to prove Theorems 1.2 and 1.3.

Proposition 4.2. *Let $T = (V, A)$ be an n-tournament and let C be a subset of V such that $2 \leq |C| \leq n - 1$. Given a decomposability graph $G = (V, E)$ for C, the following statements hold*

- *if $n - \delta_T \leq |C|$, then $s_C(T) \geq \delta_T$;*
- *if $|C| \leq \sigma_T$, then $s_C(T) \geq \sigma_T$.*

Proof. Before showing the first assertion, we establish

$$|E| \geq (n - |C|)(|C| - (n - 1 - \delta_T)). \tag{4.2}$$

Let $x \in V \smallsetminus C$. By the second assertion of Lemma 4.1

$$\begin{aligned}\delta_G(x) &= \min(\left|N_T^-(x) \cap C\right|, \left|N_T^+(x) \cap C\right|) \\ &= |C| - \max(\left|N_T^-(x) \cap C\right|, \left|N_T^+(x) \cap C\right|).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\delta_G(x) &\geq |C| - \max(\left|N_T^-(x)\right|, \left|N_T^+(x)\right|) \\ &\geq (|C| - (n - 1 - \delta_T)).\end{aligned} \tag{4.3}$$

Since G is bipartite with bipartition $\{C, V \setminus C\}$, we have

$$|E| = \sum_{x \in V \smallsetminus C} \delta_G(x).$$

It follows from (4.3) that

$$\begin{aligned}|E| &\geq |V \setminus C|\,(|C| - (n-1-\delta_T)) \\ &\geq (n-|C|)(|C| - (n-1-\delta_T)).\end{aligned}$$

Thus, (4.2) holds. Moreover, we have

$$(n-|C|)(|C| - (n-1-\delta_T)) - \delta_T = (n-1-|C|)(|C| - (n-\delta_T)).$$

Now, to prove the first assertion, suppose that $n - \delta_T \leq |C|$. We obtain

$$(n-1-|C|)(|C| - (n-\delta_T) \geq 0,$$

and hence

$$(n-|C|)(|C| - (n-1-\delta_T)) \geq \delta_T.$$

It follows that $s_C(T) = |E| \geq \delta_T$.

Before showing the second assertion, we establish

$$|E| \geq \frac{|C|}{2}(2 - |C| + \sigma_T). \tag{4.4}$$

Consider two vertices $x \neq y \in C$. For convenience, denote by $\mathcal{S}_T(x, y)$ the set of separators of $\{x, y\}$. Clearly, we have $(\mathcal{S}_T(x, y) \setminus C) \subseteq N_G(x) \cup N_G(y)$. It follows that

$$\delta_G(x) + \delta_G(y) \geq |\mathcal{S}_T(x, y) \setminus C| \geq \sigma_T(x, y) - (|C| - 2).$$

Consequently, we obtain

$$\delta_G(x) + \delta_G(y) \geq \sigma_T - |C| + 2. \tag{4.5}$$

Furthermore, observe that

$$\sum_{\{x,y\} \in \binom{C}{2}} (\delta_G(x) + \delta_G(y)) = (|C| - 1) \sum_{x \in C} \delta_G(x).$$

It follows from (4.5) that

$$(|C| - 1) \sum_{x \in C} \delta_G(x) \geq \binom{|C|}{2}(2 - |C| + \sigma_T).$$

Therefore, we have

$$\sum_{x \in C} \delta_G(x) \geq \frac{|C|}{2}(2 - |C| + \sigma_T).$$

Since G is bipartite with bipartition $\{C, V \setminus C\}$, we have

$$|E| = \sum_{x \in C} \delta_G(x).$$

We obtain

$$|E| \geq \frac{|C|}{2}(2 - |C| + \sigma_T),$$

so (4.4) holds.

Finally, to prove the second assertion, suppose that $|C| \leq \sigma_T$. We obtain

$$\frac{|C|}{2}(2 - |C| + \sigma_T) \geq \sigma_T.$$

Since $s_C(T) = |E|$, it follows from (4.4) that $s_C(T) \geq \sigma_T$. □

Proof of Theorem 1.2. Let $l \in \{1, 2, 3\}$. Consider a tournament R from T by removing l vertices $v_1, \ldots, v_l$. Set $V' := V \setminus \{v_1, \ldots, v_l\}$. It follows from Theorem 1.1 that $s(R) \leq (2k+1) - l$. It remains to show that $s(R) \geq (2k+1) - l$. By (4.1), it suffices to verify that $s_C(R) \geq (2k+1) - l$ for every subset C of V' such that

$$2 \leq |C| \leq (4k+2) - l.$$

Let $C \subseteq V'$ such that

$$2 \leq |C| \leq (4k+2) - l.$$

We distinguish the following three cases.

Case 1: Suppose that $2 \leq |C| \leq (2k+1) - l$.

Since T is doubly regular, it follows from Proposition 2.4 that $\sigma_T = 2k+1$. Therefore, $\sigma_R \geq (2k+1) - l$. Since

$$2 \leq |C| \leq (2k+1) - l, \quad \sigma_R \geq |C|.$$

It follows from Proposition 4.2 that $s_C(R) \geq \sigma_R$, and hence $s_C(R) \geq (2k+1) - l$.

Case 2: Suppose that $2k+2 \leq |C| \leq (4k+2) - l$.

Since T is doubly regular, it follows from Proposition 2.4 that T is regular. Thus, $\delta_T = 2k+1$. It follows that $\delta_R \geq (2k+1) - l$. Since

$$2k+2 \leq |C| \leq (4k+2) - l,$$

we obtain $|C| + \delta_R \geq (4k+3) - l$. It follows from Proposition 4.2 that $s_C(R) \geq \delta_R$, and hence $s_C(R) \geq (2k+1) - l$.

Case 3: $(2k+2) - l \leq |C| \leq 2k+1$.

Let $G = (E', V')$ be a decomposability graph for C. We verify that

$$|\{x \in V' \setminus C : \delta_G(x) \neq 0\}| \geq |V' \setminus C| - 1. \tag{4.6}$$

Otherwise, there exist $x \neq y \in V' \setminus C$ such that $\delta_G(x) = \delta_G(y) = 0$. It follows from the second assertion of Lemma 4.1 applied to R that C is contained in one of the following intersections:

$$(N_R^-(x) \cap N_R^+(y)), \qquad (N_R^-(x) \cap N_R^-(y)),$$

$$(N_R^+(x) \cap N_R^+(y)), \quad \text{or} \quad (N_R^+(x) \cap N_R^-(y)).$$

Thus, C is contained in

$$(N_T^-(x) \cap N_T^+(y)), \qquad (N_T^-(x) \cap N_T^-(y)),$$

$$(N_T^+(x) \cap N_T^+(y)), \quad \text{or} \quad (N_T^+(x) \cap N_T^-(y)).$$

It follows from Proposition 2.4 that $|C| \leq k+1$, which contradicts $|C| \geq (2k+2)-l$ because $k \geq l$. Consequently, (4.6) holds. Since G is bipartite with bipartition $\{C, V' \smallsetminus C\}$, we have

$$\left|E'\right| = \sum_{x \in V' \smallsetminus C} \delta_G(x).$$

Since $|E'| = s_C(R)$, we obtain

$$\begin{aligned} s_C(R) &= \sum_{x \in V' \smallsetminus C} \delta_G(x) \\ &\geq \left|V' \smallsetminus C\right| - 1 \quad \text{(by (4.6))} \\ &\geq (2k+1) - l \quad (\text{because } |C| \leq 2k+1). \end{aligned}$$

□

5. Proof of Theorem 1.3

If a tournament T is obtained from a doubly regular $(4k+3)$-tournament by deleting one vertex, then T is near-regular and it follows from Proposition 2.4 that

(C1) if $x, y \in V_{\text{even}}$ (see Notation 2.2) or $x, y \in V_{\text{odd}}$, then $\sigma_T(x,y) = 2k+1$.
(C2) if $x \in V_{\text{even}}$ and $y \in V_{\text{odd}}$, then $\sigma_T(x,y) = 2k$.

Conversely, we have the following proposition.

Proposition 5.1. *Let $T = (V, A)$ be a near-regular tournament of order $4k+2$. If T satisfies (C1) and (C2), then the tournament U obtained from T by adding a vertex ω which dominates V_{odd} and is dominated by V_{even} is doubly regular.*

The proof of this proposition uses the following lemma.

Lemma 5.2. *Under the notation and conditions of Proposition 5.1, for every $x, y \in V$ such that $x \to y$, we have*

- *if $x, y \in V_{\text{odd}}$, then*
$$\left|N_T^-(x) \cap N_T^+(y)\right| = k+1 \text{ and } \left|N_T^+(x) \cap N_T^-(y)\right| = k;$$
- *if $x, y \in V_{\text{even}}$, then*
$$\left|N_T^-(x) \cap N_T^+(y)\right| = k+1 \text{ and } \left|N_T^+(x) \cap N_T^-(y)\right| = k;$$
- *if $x \in V_{\text{odd}}$ and $y \in V_{\text{even}}$, then*
$$\left|N_T^-(x) \cap N_T^+(y)\right| = k \text{ and } \left|N_T^+(x) \cap N_T^-(y)\right| = k;$$
- *if $x \in V_{\text{even}}$ and $y \in V_{\text{odd}}$, then*
$$\left|N_T^-(x) \cap N_T^+(y)\right| = k+1 \text{ and } \left|N_T^+(x) \cap N_T^-(y)\right| = k-1.$$

Proof. We have

$$(5.1)\qquad \begin{cases} \left|N_T^-(x)\cap N_T^-(y)\right| + \left|N_T^-(x)\cap N_T^+(y)\right| = \left|N_T^-(x)\right| \\ \text{and} \\ \left|N_T^+(x)\cap N_T^+(y)\right| + \left|N_T^-(x)\cap N_T^+(y)\right| = \left|N_T^+(y)\right|. \end{cases}$$

By using Lemma 2.3, we obtain

$$(5.2)\qquad \left|N_T^-(x)\cap N_T^+(y)\right| = \frac{1}{2}\left(\left|N_T^-(x)\right| + \left|N_T^+(y)\right| - 4k + \sigma_T(x,y)\right).$$

Using Assertions (C1) and (C2), we obtain the desired values of

$$|N_T^-(x)\cap N_T^+(y)|.$$

Then, $|N_T^+(x)\cap N_T^-(y)|$ follows immediately because

$$|N_T^+(x)\cap N_T^-(y)| = \sigma(x,y) - |N_T^-(x)\cap N_T^+(y)|.$$

□

Proof of Proposition 5.1. Clearly, U is regular. Furthermore, by Lemma 2.3,

$$\delta_U^+(x,y) = \frac{4k - \sigma_U(x,y)+1}{2}$$

for distinct $x, y \in V\cup\{\omega\}$. Therefore, U is doubly regular if and only if $\sigma_U(x,y) = 2k+1$ for every $x,y\in V\cup\{\omega\}$. This equality follows directly from (C1) and (C2) when $x,y\in V$. Hence, it remains to prove that

$$(5.3)\qquad \sigma_U(\omega,z) = 2k+1 \text{ for every } z\in V.$$

Consider $z\in V$. It is not difficult to see that

$$\sigma_U(\omega,z) = \left|N_T^+(z)\cap V_{\text{even}}\right| + \left|N_T^-(z)\cap V_{\text{odd}}\right| \text{ (see Notation 2.2).}$$

Let

$$A_{\text{odd}} := (N_T^+(z)\cap V_{\text{odd}}),\qquad A_{\text{even}} := (N_T^+(z)\cap V_{\text{even}}),$$
$$B_{\text{odd}} := (N_T^-(z)\cap V_{\text{odd}}),\quad \text{and}\quad B_{\text{even}} := (N_T^-(z)\cap V_{\text{even}}).$$

We determine $|A_{\text{odd}}|$, $|A_{\text{even}}|$, $|B_{\text{odd}}|$, and $|B_{\text{even}}|$ as follows.

To begin, suppose that $z\in V_{\text{odd}}$. By counting the number of arcs from $N_T^+(z)$ to $N_T^-(z)$ in two ways, we get

$$\sum_{t\in A_{\text{odd}}}\left|N_T^-(z)\cap N_T^+(t)\right| + \sum_{t\in A_{\text{even}}}\left|N_T^-(z)\cap N_T^+(t)\right|$$
$$= \sum_{t\in B_{\text{odd}}}\left|N_T^-(t)\cap N_T^+(z)\right| + \sum_{t\in B_{\text{even}}}\left|N_T^-(t)\cap N_T^+(z)\right|.$$

It follows from Lemma 5.2 that

$$(k+1)\left|A_{\text{odd}}\right| + k\left|A_{\text{even}}\right| = (k+1)(|B_{\text{odd}}| + |B_{\text{even}}|).$$

Since $z\in V_{\text{odd}}$, we have

$$|A_{\text{odd}}| + |A_{\text{even}}| = 2k+1,\qquad |B_{\text{odd}}| + |B_{\text{even}}| = 2k,$$
$$|A_{\text{odd}}| + |B_{\text{odd}}| = 2k,\quad \text{and}\quad |A_{\text{even}}| + |B_{\text{even}}| = 2k+1.$$

It follows that $|A_{\text{odd}}| = k$, $|B_{\text{odd}}| = k$, $|B_{\text{even}}| = k$, and $|A_{\text{even}}| = k+1$.

Similarly, if $z \in V_{\text{even}}$, then $|A_{\text{odd}}| = k$, $|B_{\text{odd}}| = k+1$, $|B_{\text{even}}| = k$, and $|A_{\text{even}}| = k$.

Consequently, (5.3) holds whatever the parity of $\delta_T^+(z)$. □

Proof of Theorem 1.3. Given $k \geq 1$, consider a tournament T, with $4k+2$ vertices, such that $s(T) = 2k$. By Proposition 3.1, $\delta_T \geq 2k$. Thus, T is near-regular. We conclude by applying Proposition 5.1. Therefore, it suffices to verify that (C1) and (C2) are satisfied.

By Proposition 3.1, $\sigma_T(x,y) \geq 2k$ for distinct $x, y \in V$. Moreover, it follows from Lemma 2.3 that if $x, y \in V_{\text{even}}$ or $x, y \in V_{\text{odd}}$ (see Notation 2.2), then $\sigma_T(x,y)$ is odd and hence $\sigma_T(x,y) \geq 2k+1$.

Lastly, seeking a contradiction, suppose that (C1) or (C2) are not satisfied. One of the following situations occurs

- there are distinct $x, y \in V_{\text{even}}$ such that $\sigma_T(x,y) > 2k+1$,
- there are distinct $x, y \in V_{\text{odd}}$ such that $\sigma_T(x,y) > 2k+1$,
- there are $x \in V_{\text{even}}$ and $y \in V_{\text{odd}}$ such that $\sigma_T(x,y) > 2k$.

We obtain

$$\begin{aligned}\sum_{\{x,y\} \in \binom{V}{2}} \sigma_T(x,y) &> (2k+1)\binom{|V_{\text{even}}|}{2} + (2k+1)\binom{|V_{\text{odd}}|}{2} + 2k\,|V_{\text{even}}|\,|V_{\text{odd}}| \\ &= 4k(2k+1)^2,\end{aligned}$$

which contradicts (2.2). Consequently, (C1) and (C2) are satisfied. □

6. Concluding remarks

1. An n-tournament with $n = 4k+1$ is called *near-homogeneous* [19] if every unordered pair of its vertices belongs to k or $(k+1)$ 3-cycles. The existence of near-homogeneous tournaments is discussed in [19], [1], and [14]. For $n \equiv 1 \pmod 4$ or $n \equiv 0 \pmod 4$, the n-tournaments given in Theorem 1.2 are not the only ones with a maximal simplicity index. Indeed, let T be a near-homogeneous tournament T with $4k+1$ vertices. By adapting the proof of Theorem 1.2, we can verify that $s(T) = 2k-1$. Moreover, by removing one vertex from T, we obtain a $(4k)$-tournament whose simplicity index is $2k-2$. Consequently, an analogue of Theorem 1.3 does not exist when $l = 2$ or 3.

2. The score vector of a n-tournament T is the ordered sequence of the scores of T listed in a nondecreasing order. Kirkland [11] proved that the reversal index of an n-tournament T is equal to

$$\min\left\{\sum_{i=1}^{j} s_i - \binom{j}{2} : 1 \leq j \leq n\right\},$$

where $(s_1, s_2, \ldots, s_n)$ is the score vector of T.

An equivalent form of this result was obtained earlier by Li and Huang [13]. As a consequence, two tournaments with the same score vector have the same reversal index. This fact is not true for the simplicity index. Indeed, for an odd number n, consider the n-tournament R_n whose vertex set is the additive group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ of integers modulo n, such that i dominates j if and only if $i - j \in \{1, \dots, (n-1)/2\}$. It is not difficult to verify that the tournament R_n is regular and simple. Moreover, by reversing the arc $(0, (n-1)/2)$, we obtain a nonsimple tournament. Hence, the simplicity index of R_n is 1. If n is prime and $n \equiv 3 \pmod 4$, the Paley tournament P_n is also regular but its simplicity index is $(n-1)/2$.

Let T be an n-tournament with vertex set $\{v_1, \dots, v_n\}$. The sequences $L_1 = (\delta^+_T(v_i))_{1 \leq i \leq n}$ and $L_2 = (\delta^+_T(v_i, v_j))_{1 \leq i < j \leq n}$ are frequently used in our study of the simplicity index. It is natural to ask whether the simplicity index of T can be expressed in terms of L_1 and L_2.

References

1. A. Astié-Vidal and V. Dugat, *Near-homogeneous tournaments and permutation groups*, Discrete mathematics **102** (1992), no. 2, 111–120.
2. H. Belkhechine and C. Ben Salha, *Decomposability and co-modular indices of tournaments*, Discrete Mathematics **344** (2021), no. 4, 112272.
3. E. Brown and K. B. Reid, *Doubly regular tournaments are equivalent to skew hadamard matrices*, Journal of Combinatorial Theory, Series A **12** (1972), no. 3, 332–338.
4. A. Cournier and M. Habib, *An efficient algorithm to recognize prime undirected graphs*, Proceedings of the 18th International Workshop on Graph-Theoretic Concepts in Computer Science, 1992, pp. 212–224.
5. A. Ehrenfeucht and G. Rozenberg, *Primitivity is hereditary for 2-structures*, Theoretical Computer Science **70** (1990), no. 3, 343–358.
6. P. Erdős, E. Fried, A. Hajnal, and E. C. Milner, *Some remarks on simple tournaments*, Algebra Universalis **2** (1972), no. 1, 238–245.
7. P. Erdős and J. W. Moon, *On sets of consistent arcs in a tournament*, Canad. Math. Bull **8** (1965), 269–271.
8. E. Fried and H. Lakser, *Simple tournaments*, Notices of the American Mathematical Society, vol. 18(2), 1971, p. 395.
9. T. Gallai, *Transitiv orientierbare graphen*, Acta Mathematica Academiae Scientiarum Hungarica **18** (1967), no. 1-2, 25–66.
10. P. Ille, *Indecomposable graphs*, Discrete Mathematics **173** (1997), no. 1-3, 71–78.
11. S. Kirkland, *A reversal index for tournament matrices*, Linear and Multilinear Algebra **34** (1993), no. 3-4, 343–351.
12. C. Koukouvinos and S. Stylianou, *On skew-hadamard matrices*, Discrete Mathematics **308** (2008), no. 13, 2723–2731.
13. J. Li and G. Huang, *Score vectors of tournaments*, Annals of the New York Academy of Sciences **576** (1989), no. 1, 323–327.
14. A. Moukouelle, *Construction d'une nouvelle classe de tournois presqu'homogènes*, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics **327** (1998), no. 11, 913–916.
15. V. Müller and J. Pelant, *On strongly homogeneous tournaments*, Czechoslovak Mathematical Journal **24** (1974), no. 3, 378–391.
16. S. Satake, *A constructive solution to a problem of ranking tournaments*, Discrete Mathematics **342** (2019), no. 12, 111613.
17. J. H. Schmerl and W. T. Trotter, *Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures*, Discrete Mathematics **113** (1993), no. 1-3, 191–205.

18. P. Slater, *Inconsistencies in a schedule of paired comparisons*, Biometrika **48** (1961), no. 3/4, 303–312.
19. C. Tabib, *Caractérisation des tournois presqu'homogènes*, Annals of Discrete Mathematics, vol. 8, Elsevier, 1980, pp. 77–82.
20. J. Wallis, *Some (1, -1) matrices*, Journal of Combinatorial Theory, Series B **10** (1971), no. 1, 1–11.

Laboratoire de Mathématiques Fondamentales et Appliquées, Faculté des Sciences Aïn Chock, Hassan II University of Casablanca, Morocco
E-mail address: `aboussairi@hotmail.com`

Laboratoire de Mathématiques Fondamentales et Appliquées, Faculté des Sciences Aïn Chock, Hassan II University of Casablanca, Morocco
E-mail address: `s.lakhlifi1@gmail.com`

Laboratoire de Mathématiques Fondamentales et Appliquées, Faculté des Sciences Aïn Chock, Hassan II University of Casablanca, Morocco
E-mail address: `italbaoui@gmail.com`