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DIOPHANTINE EQUATION WITH BALANCING-LIKE SEQUENCES ASSOCIATED WITH THE SHOREY–TIJDEMAN-TYPE PROBLEM

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Abstract. Let $\{x_n\}_{n\geq 0}$ be the balancing-like sequence defined by

$$x_{n+1} = Ax_n - x_{n-1}$$

with initial terms $x_0 = 0, x_1 = 1$ for $A \geq 3$. In this study, we demonstrate how to find all the solutions of the Diophantine equation,

$$\sum_{i=1}^{3} C_i x_{n_i} = \sum_{i=4}^{6} C_i x_{n_i}$$

in fixed integers $A \geq 3$, $n_1 > n_2 > n_3 \geq 0, n_4 > n_5 > n_6 \geq 0$, and $C_1 x_{n_1} \neq C_4 x_{n_4}$, where $C_i; 1 \leq i \leq 6$ are given integers satisfying $C_1 C_2 C_3 \neq 0$.

1. Introduction

A balancing number B is a natural number satisfying the Diophantine equation

$$\sum_{j=1}^{B-1} j = \sum_{k=B+1}^{R} k,$$

for some natural number R [1]. If B is a balancing number, then $8B^2+1$ is a perfect square. The n-th balancing number is denoted by B_n. The balancing numbers satisfy the binary recurrence $B_{n+1} = 6B_n - B_{n-1}, B_0 = 0, B_1 = 1$ which holds for $n \geq 1$. The balancing sequence has been studied extensively and generalized in many ways [4, 5, 6, 7]. As a generalization of the balancing sequence, Panda and Rout [5] introduced a family of binary recurrences defined by

$$x_{n+1} = Ax_n - x_{n-1},\ x_0 = 0,\ x_1 = 1,$$

for any fixed integer $A \geq 3$. Subsequently, these sequences were called balancing-like sequences $\{x_n\}_{n\geq 0}$ since the particular case corresponding to

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$A = 6$ coincides with the balancing sequence. When $A = 2$, the binary sequence $x_{n+1} = 2x_n - x_{n-1}$ with $x_0 = 0, x_1 = 1$ generates the natural numbers. The sequence $\{x_n\}_{n\geq 0}$ behaves like the sequence of natural numbers, and for this reason, Khan and Kwong [3] called these sequences generalized natural number sequences. For each n, $Dx_n^2 + 1$ is a perfect rational square, where $D = \frac{A^2-4}{4}$, is a perfect square and its positive square root is called a Lucas-balancing-like number [5]. They have also noted that all the balancing-like sequences are strong divisibility sequences. Further, different associate sequences of a balancing-like sequence have been studied by Panda and Pradhan [6]. The Binet formula of a balancing-like sequence $\{x_n\}_{n\geq 0}$ is given by

$$x_n = \frac{\gamma^n - \delta^n}{\sqrt{A^2-4}},\ n = 1, 2, \ldots,$$

where $\gamma := \frac{A+\sqrt{A^2-4}}{2}$ and $\delta := \frac{A-\sqrt{A^2-4}}{2}$ are the roots of the characteristic equation $X^2 - AX + 1 = 0$. It is easy to see that $\gamma = \delta^{-1}$. Balancing-like sequences satisfy the following divisibility property.

Lemma 1.1 ([5, Theorem 2.8])**.** *For all* $m, n \in \mathbb{N}$, $\gcd(x_m, x_n) = x_{\gcd(m,n)}$.

In order to solve the main result, the following lemma is crucial.

Lemma 1.2 ([8, Lemma 2.2])**.** *For every positive integer* $n \geq 1$,

$$\gamma^{n-1} \leq x_n \leq \gamma^n.$$

Sahukar and Panda [9] dealt with the Brocard–Ramanujan-type equations $x_{n_1}x_{n_2}\cdots x_{n_k} \pm 1 = x_m$ or y_m or y_m^2 where $\{x_n\}_{n\geq 0}$ and $\{y_m\}_{m\geq 0}$ are either balancing-like sequences or associated balancing-like sequences. Subsequently, they [10] showed that each balancing-like sequence has at most three terms that are one away from perfect squares.

Shorey and Tijdeman [11] proved under suitable conditions, that the equation $Ax^m + By^m = Cx^n + Dy^n$ implies that $\max\{n, m\}$ is bounded by a computable constant depending only on A, B, C, D, where $A, B, C, D, m, n > 0$. Recently, Ddamulira et al. [2] studied a variation of the equation $Ax^m + By^m = Cx^n + Dy^n$ with the terms of the Lucas sequence $\{U_n\}_{n\geq 0}$. They showed that there are two types of solutions of the Diophantine equation $AU_n + BU_m = CU_{n_1} + DU_{m_1}$, sporadic solutions and parametric solutions.

Motivated by the above literature, we examine a variation of the Shorey and Tijdeman equation related to balancing-like sequences. We study the Diophantine equation

$$\sum_{i=1}^{3} C_i x_{n_i} = \sum_{i=4}^{6} C_i x_{n_i} \tag{1.1}$$

with $n_1 > n_2 > n_3 \geq 0, n_4 > n_5 > n_6 \geq 0$, and $C_1x_{n_1} \neq C_4x_{n_4}$, where $\{x_n\}_{n\geq 0}$ is a balancing-like sequence and $C_i; 1 \leq i \leq 6$ are given integers such that $C_1C_2C_3 \neq 0$. More precisely, we have the following result.

Theorem 1.3. *Let* $X := \max\{|C_i|; 1 \le i \le 6\}$ *and* $\psi := (3+\sqrt{5})/2$ *be the smallest possible* γ. *Relabel the variables* $(n_1, n_2, n_3, n_4, n_5, n_6)$ *as* $(m_1, m_2, m_3, m_4, m_5, m_6)$, *where* $m_1 \ge m_2 \ge m_3 \ge m_4 \ge m_5 \ge m_6$. *If* $n_1 = n_4$, *we rewrite the Diophantine equation* (1.1) *as*

$$(C_1 - C_4)x_{n_1} + C_2 x_{n_2} + C_3 x_{n_3} = C_5 x_{n_4} + C_6 x_{n_5},$$

and change $(C_1, C_2, C_3, C_4, C_5, C_6)$ *to* $(C_1 - C_4, C_2, C_3, C_5, C_6, 0)$. *Thus,* $m_1 > m_2$. *Furthermore, we change the sign of some of the coefficients* $(C_1, C_2, C_3, C_4, C_5, C_6)$ *so that the Diophantine equation* (1.1) *becomes*

$$\sum_{i=1}^{6} A_i x_{m_i} = 0. \tag{1.2}$$

Assume $A \le 308X$ *for any fixed integer* $A \ge 3$ *of the characteristic equation* $X^2 - AX + 1 = 0$. *Then, the solutions of the Diophantine equation* (1.2) *are of two types:*

(i) *Sporadic ones. These are finitely many and they satisfy:*

$$m_6 \le \frac{\log(12X)}{\log\psi}, m_5 \le \frac{\log(1000X^3)}{\log\psi}, m_4 \le \frac{\log(123000X^5)}{\log\psi},$$
$$m_3 \le \frac{\log(17700000X^7)}{\log\psi}, m_2 \le \frac{\log(4530000000X^9)}{\log\psi},$$
$$m_1 \le \frac{\log(99660000000X^{10})}{\log\psi}.$$

(ii) *Parametric ones. These are one of the two forms:*

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (m_5 + l, m_5 + k, m_5 + j, m_5 + i, m_5, 0),$$

where

$$l \le \frac{\log(104000000X^7)}{\log\psi},\ k \le \frac{\log(4720000X^6)}{\log\psi},$$
$$j \le \frac{\log(18500X^4)}{\log\psi} \text{ and } i \le \frac{\log(130X^2)}{\log\psi},$$

and γ *is a root of* $A_1X^l + A_2X^k + A_3X^j + A_4X^i + A_5 = 0$, *or of the form*

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (m_6 + m, m_6 + l, m_6 + k, m_6 + j, m_6 + i, m_6),$$

where

$$m \le \frac{\log(8305000000X^9)}{\log\psi},\ l \le \frac{\log(377500000X^8)}{\log\psi},\ k \le \frac{\log(1485000X^6)}{\log\psi},$$
$$j \le \frac{\log(10300X^4)}{\log\psi} \text{ and } i \le \frac{\log(80X^2)}{\log\psi},$$

and γ *is a root of*

$$A_1X^m + A_2X^l + A_3X^k + A_4X^j + A_5X^i + A_6 = 0.$$

2. Main result

To achieve the objective of this paper, we need the following result.

Lemma 2.1. *Let $C_i; 1 \leq i \leq 6$ are given integers such that $C_1C_2C_3 \neq 0$ and (1.1) holds. Then $A < 308X$, where $X := \max\{|C_i|; 1 \leq i \leq 6\}$.*

Proof. Case 1. If $C_3 = C_6 = 0$, then (1.1) becomes $C_1x_{n_1} + C_2x_{n_2} = C_4x_{n_4} + C_5x_{n_5}$ for any $n_3, n_6 \geq 0$. We have the following subcases for it.
Subcase 1. If $C_4 = C_5 = 0$, then $C_1x_{n_1} = -C_2x_{n_2}$ for any $n_4 > n_5 \geq 0$. Since $C_1x_{n_1} \neq C_4x_{n_4}$ and $C_1C_2 \neq 0$, we get both n_1 and n_2 are nonzero. Thus from Lemma 1.1 and by taking $n_2 \neq 0$, we have $\frac{x_{n_1}}{x_d}$ divides C_2, where $d := \gcd(n_1, n_2)$. For $l \geq 2$, let $ld := n_1$. If $d = 1$, then $\frac{x_{n_1}}{x_d} = \frac{x_l}{x_1} = x_l \geq x_2 = A$, so $A \leq X$. For $d \geq 2$, we have $\frac{x_{n_1}}{x_d} = \frac{\gamma^{ld}-\delta^{ld}}{\gamma^d-\delta^d}$. Now, we can show that

$$\frac{\gamma^{ld} - \delta^{ld}}{\gamma^d - \delta^d} > \gamma,$$

which is equivalent to $\gamma^{ld} - \gamma^{d+1} > \delta^{ld} - \delta^{d-1}$. Since $d \geq 2$ and $|\delta^{d-1}| < 1$, therefore it is sufficient to show that $\gamma^{2d} - \gamma^{d+1} > 29$. The left-hand side is

$$\begin{aligned} \gamma^{2d} - \gamma^{d+1} &= \gamma^{d+1}(\gamma^{d-1} - 1) \\ &\geq \gamma^{d+1}(\gamma - 1). \end{aligned}$$

The smallest possible γ is $\psi = \frac{3+\sqrt{5}}{2}$ (for $A = 3$) and $\psi^{d+1}(\psi - 1) \geq \psi^3(\psi - 1) > 29$. Hence, $\gamma < \frac{x_{ld}}{x_d} \leq X$, which gives $A = \gamma + \delta < \gamma < X$. By virtue of Lemma 1.2, we have $x_{n_1} \geq \gamma^{n_1-1}$ and $x_d \leq \gamma^d$, so $\frac{x_{n_1}}{x_d} \geq \frac{\gamma^{n_1-1}}{\gamma^d} = \gamma^{n_1-d-1}$. Since d is a proper divisor of n_1 and $d \geq 2$, we get $d \leq \frac{n_1}{2}$. Therefore we obtain $\frac{x_{n_1}}{x_d} \geq \gamma^{\frac{n_1}{2}-1}$. Since $\frac{x_{n_1}}{x_d}$ divides C_2, we get

$$\gamma^{\frac{n_1}{2}-1} \leq |C_2| \leq X.$$

Taking the logarithm on both sides of the above equation leads to

$$n_1 \leq 2 + 2\frac{\log X}{\log \gamma}.$$

Since $\gamma \geq \psi$, we have

$$0 < n_2 < n_1 \leq 2 + 2\frac{\log X}{\log \psi}. \tag{2.1}$$

Subcase 2. If one of C_4, C_5 is nonzero and the other is zero. Therefore we assume that $C_4 \neq 0, C_5 = 0$ and $n_4 \neq 0$. Then we have

$$C_1x_{n_1} + C_2x_{n_2} = C_4x_{n_4}.$$

If $n_1 = n_4$, then $(C_1-C_4)x_{n_1} = -C_2x_{n_2}$, so from *Subcase 1*, we get the bound $A \leq X$. If $n_1 \neq n_4$ and $n_2 = 0$, then again from *Subcase 1*, we obtain the bound $A \leq X$. If $n_1 \neq n_4$ and $n_2 \neq 0$, then $C_1x_{n_1} + C_2x_{n_2} = C_4x_{n_4}$. Since $n_1 \neq n_4$, we have to switch C_1 with C_4. If needed, assume $n_1 = \max\{n_1, n_4\}$, then $n_1 > n_4$. We rename our indices (n_1, n_2, n_4) as (m_1, m_2, m_3) where $m_1 > m_2 \geq m_3$ and the coefficients C_1, C_2, C_4 as A_1, A_2, A_3 and change the signs to no more than a few of them, making our equation

$$A_1 x_{m_1} + A_2 x_{m_2} + A_3 x_{m_3} = 0. \tag{2.2}$$

Using the Binet formula of $\{x_n\}_{n\geq 0}$, we have

$$|A_1|\gamma^{m_1} = |-A_2(\gamma^{m_2} - \delta^{m_2}) - A_3(\gamma^{m_3} - \delta^{m_3}) - A_1\delta^{m_1}| < 5X\gamma^{m_2}.$$

Hence,

$$\gamma^{m_1 - m_2} < 5X. \tag{2.3}$$

Thus, since $m_1 > m_2$, we get that $A < \gamma \leq \gamma^{m_1-m_2} < 5X$. Keeping in mind that we might need to substitute $2X$ instead of X, we get the bound $A < 10X$.

Subcase 3. If both C_4 and C_5 are nonzero, then

$$C_1 x_{n_1} + C_2 x_{n_2} = C_4 x_{n_4} + C_5 x_{n_5}.$$

If $n_1 = n_4$, then $(C_1 - C_4)x_{n_1} + C_2 x_{n_2} = C_5 x_{n_5}$. If $n_2 = 0$ or $n_5 = 0$, then again from *Subcase 1*, we get the same bound $A \leq X$. If both n_2, n_5 are nonzero, then we replace $(C_1, C_2, C_4, C_5) \longrightarrow (C_1 - C_4, C_2, C_5, 0)$ and similar to the *Subcase 2*, we get the bound $A < 10X$. If $n_1 \neq n_4$ and n_2, n_5 both are nonzero, then $C_1 x_{n_1} + C_2 x_{n_2} = C_4 x_{n_4} + C_5 x_{n_5}$. Since $n_1 \neq n_4$, we have to switch C_1 with C_4. If needed, assume $n_1 = \max\{n_1, n_4\}$, then $n_1 > n_4$. We rename our indices (n_1, n_2, n_4, n_5) as (m_1, m_2, m_3, m_4) where $m_1 > m_2 \geq m_3 \geq m_4$ and the coefficients C_1, C_2, C_4, C_5 as A_1, A_2, A_3, A_4 and change the signs to no more than a few of them, making our equation

$$A_1 x_{m_1} + A_2 x_{m_2} + A_3 x_{m_3} + A_4 x_{m_4} = 0. \tag{2.4}$$

Using the Binet formula of $\{x_n\}_{n\geq 0}$, we have

$$\begin{aligned}|A_1|\gamma^{m_1} &= |-A_2(\gamma^{m_2} - \delta^{m_2}) - A_3(\gamma^{m_3} - \delta^{m_3}) - A_4(\gamma^{m_4} - \delta^{m_4}) - A_1\delta^{m_1}| \\ &< 7X\gamma^{m_2}.\end{aligned}$$

Hence,

$$\gamma^{m_1 - m_2} < 7X. \tag{2.5}$$

Thus, since $m_1 > m_2$, we get that $A < \gamma \leq \gamma^{m_1-m_2} < 7X$. Keeping in mind that we might need to substitute $2X$ instead of X, we get the bound $A < 14X$.

Case 2. If one of C_5, C_6 is nonzero and the other is zero. Assume that $C_5 \neq 0$ and $n_5 \neq 0$. If $C_6 = 0$, then

$$C_1 x_{n_1} + C_2 x_{n_2} + C_3 x_{n_3} = C_4 x_{n_4} + C_5 x_{n_5}.$$

If $n_1 = n_4$, then $(C_1 - C_4)x_{n_1} + C_2 x_{n_2} + C_3 x_{n_3} = C_5 x_{n_5}$. So using *Case 1*, we get the bound $A < 14X$. Keeping in mind that we might need to substitute $2X$ instead of X, we get the bound $A < 28X$. Thus, we may assume that $n_1 \neq n_4$, then switch C_1 with C_4. Assume that $n_1 = \max\{n_1, n_4\}$. Therefore $n_1 > n_4$. We rename our indices $(n_1, n_2, n_3, n_4, n_5)$ as $(m_1, m_2, m_3, m_4, m_5)$

where $m_1 > m_2 > m_3 \geq m_4 \geq m_5$, and the coefficients C_1, C_2, C_3, C_4, C_5 as A_1, A_2, A_3, A_4, A_5 and change the signs to no more than a few of them, making our equation

$$A_1 x_{m_1} + A_2 x_{m_2} + A_3 x_{m_3} + A_4 x_{m_4} + A_5 x_{m_5} = 0. \tag{2.6}$$

Using the Binet formula of $\{x_n\}_{n\geq 0}$, we have

$$\begin{aligned}|A_1|\gamma^{m_1} = &| - A_2(\gamma^{m_2} - \delta^{m_2}) - A_3(\gamma^{m_3} - \delta^{m_3}) - A_4(\gamma^{m_4} - \delta^{m_4}) \\ &- A_5(\gamma^{m_5} - \delta^{m_5}) - A_1\delta^{n_1}| < 9X\gamma^{m_2}.\end{aligned}$$

Hence,

$$\gamma^{m_1 - m_2} < 9X. \tag{2.7}$$

Thus, since $m_1 > m_2$, we get that $A < \gamma \leq \gamma^{m_1-m_2} < 9X$. Keeping in mind that we might need to substitute $14X$ instead of X, we get the bound $A < 126X$.

Case 3. If both C_5, C_6 are nonzero. In this case, we have

$$C_1 x_{n_1} + C_2 x_{n_2} + C_3 x_{n_3} = C_4 x_{n_4} + C_5 x_{n_5} + C_6 x_{n_6}.$$

If $n_1 = n_4$, then $(C_1 - C_4)x_{n_1} + C_2 x_{n_2} + C_3 x_{n_3} = C_5 x_{n_5} + C_6 x_{n_6}$. If $n_2 = 0$ or $n_3 = 0$ or $n_5 = 0$ or $n_6 = 0$, then in all the cases, we use *Case 1* and get the bound $A \leq 28X$. Otherwise, we replace $(C_1, C_2, C_3, C_4, C_5, C_6) \longrightarrow (C_1 - C_4, C_2, C_3, C_5, C_6, 0)$. The only change is that $28X$ is replaced with $56X$. Thus, we may assume that $n_1 \neq n_4$, then switch C_1 with C_4. Suppose that $n_1 = \max\{n_1, n_4\}$. Therefore $n_1 > n_4$. We rename our indices

$$(n_1, n_2, n_3, n_4, n_5, n_6) \text{ as } (m_1, m_2, m_3, m_4, m_5, m_6)$$

where $m_1 > m_2 > m_3 \geq m_4 \geq m_5 \geq m_6$ and the coefficients $C_1, C_2, C_3, C_4, C_5, C_6$ as $A_1, A_2, A_3, A_4, A_5, A_6$ and change the signs to no more than a few of them, making our equation

$$A_1 x_{m_1} + A_2 x_{m_2} + A_3 x_{m_3} + A_4 x_{m_4} + A_5 x_{m_5} + A_6 x_{m_6} = 0. \tag{2.8}$$

Using the Binet formula of $\{x_n\}_{n\geq 0}$, we have

$$\begin{aligned}|A_1|\gamma^{m_1} = &| - A_2(\gamma^{m_2} - \delta^{m_2}) - A_3(\gamma^{m_3} - \delta^{m_3}) - A_4(\gamma^{m_4} - \delta^{m_4}) \\ &- A_5(\gamma^{m_5} - \delta^{m_5}) - A_6(\gamma^{m_6} - \delta^{m_6}) - A_1\delta^{n_1}| < 11X\gamma^{m_2}.\end{aligned}$$

Hence

$$\gamma^{m_1 - m_2} < 11X. \tag{2.9}$$

Thus, since $m_1 > m_2$, we get that $A < \gamma \leq \gamma^{m_1-m_2} < 11X$. Keeping in mind that we might need to substitute $28X$ instead of X, we get the desired bound which is $A < 308X$. □

3. Proof of Theorem 1.3

From the above lemma we establish A is bounded. Now, we will show that for all A, the equation has finitely many solutions. To do this, we consider the case where both C_5 and C_6 are nonzero. In the case where $n_1 = n_4$, by substituting $(C_1, C_2, C_3, C_4, C_5, C_6) \longrightarrow (C_1 - C_4, C_2, C_3, C_5, C_6, 0)$, we get the bound of $A < 56X$. On the other hand, if $n_1 \neq n_4$, the bound becomes $A < 308X$. Therefore, we assume that $m_1 > m_2 > m_3 \geq m_4 \geq m_5 \geq m_6$ and that (2.8) holds. Consequently, estimate (2.9) is satisfied, so

$$m_1 - m_2 < \frac{\log(11X)}{\log \psi}. \tag{3.1}$$

Using the Binet formula of the balancing-like sequences in (2.8), we get

$$\begin{aligned} \left|\gamma^{m_2}\left(A_1\gamma^{m_1-m_2} + A_2\right) - \left(\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}}\right)\right| &= |-A_3(\gamma^{m_3} - \delta^{m_3}) \\ &\quad - A_4(\gamma^{m_4} - \delta^{m_4}) - A_5(\gamma^{m_5} - \delta^{m_5}) - A_6(\gamma^{m_6} - \delta^{m_6})| \\ &\leq 8X\gamma^{m_3}. \end{aligned} \tag{3.2}$$

Since $m_1 - m_2 > 0$, $A_1\gamma^{m_1-m_2} + A_2 \neq 0$. Thus

$$\left|A_1\gamma^{m_1-m_2} + A_2\right|\left|A_1\delta^{m_1-m_2} + A_2\right| \geq 1.$$

Since $|A_1\delta^{m_1-m_2} + A_2| \leq 2X$, we get $|A_1\gamma^{m_1-m_2} + A_2| \geq 1/2X$. Further

$$\left|\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}}\right| \leq \frac{2X}{\gamma^{m_2}}.$$

Hence,

$$\left|\gamma^{m_2}\left(A_1\gamma^{m_1-m_2} + A_2\right) - \left(\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}}\right)\right| \geq \frac{\gamma^{m_2}}{2X} - \frac{2X}{\gamma^{m_2}}.$$

Now, assume

$$\frac{\gamma^{m_2}}{2X} - \frac{2X}{\gamma^{m_2}} \leq \frac{\gamma^{m_2}}{8X}. \tag{3.3}$$

Then $\gamma^{2m_2} < 6X^2$, so $\gamma^{m_2} < 3X$. Hence

$$m_6 \leq m_5 \leq m_4 \leq m_3 \leq m_2 \leq \frac{\log(3X)}{\log \psi}. \tag{3.4}$$

Using (3.1), we have

$$m_1 < \frac{\log(33X^2)}{\log \psi}. \tag{3.5}$$

If (3.3) does not hold, then

$$\frac{\gamma^{m_2}}{8X} \leq \frac{\gamma^{m_2}}{2X} - \frac{2X}{\gamma^{m_2}} \leq 8X\gamma^{m_3}.$$

Then $\gamma^{m_2-m_3} \leq 64X^2$, so

$$m_2 - m_3 \leq 2\frac{\log(8X)}{\log\psi}. \tag{3.6}$$

Rewriting (2.8), we have

$$\begin{aligned}&\left|\gamma^{m_3}\left(A_1\gamma^{m_1-m_3}+A_2\gamma^{m_2-m_3}+A_3\right)-\left(\frac{A_1}{\gamma^{m_1}}+\frac{A_2}{\gamma^{m_2}}+\frac{A_3}{\gamma^{m_3}}\right)\right|\\&=|-A_4(\gamma^{m_4}-\delta^{m_4})-A_5(\gamma^{m_5}-\delta^{m_5})-A_6(\gamma^{m_6}-\delta^{m_6})|\\&\leq 6X\gamma^{m_4}. \end{aligned}\tag{3.7}$$

Assume $A_1\gamma^{m_1-m_3}+A_2\gamma^{m_2-m_3}+A_3=0$. Let $i=m_2-m_3, j=m_1-m_3$. Then

$$j \leq \frac{\log(704X^3)}{\log\psi} \text{ and } i \leq 2\frac{\log(8X)}{\log\psi}$$

are bounded. Therefore one can find all polynomials $A_1X^j+A_2X^i+A_3$ and take the polynomial which has a root γ, where γ is a quadratic unit of norm 1. Since δ is also a root of the same polynomial for these balancing-like sequences, the left-hand side of (3.7), for any m_3, is zero. This proves that $m_4=m_5=m_6=0$ as well. Further, we have

$$(m_1,m_2,m_3,m_4,m_5,m_6)=(m_3+j,m_3+i,m_3,0,0,0)$$

is a parametric family of solutions. From now on assume that $A_1\gamma^{m_1-m_3}+A_2\gamma^{m_2-m_3}+A_3\neq 0$. Thus,

$$\left|A_1\gamma^{m_1-m_3}+A_2\gamma^{m_2-m_3}+A_3\right|\left|A_1\delta^{m_1-m_3}+A_2\delta^{m_1-m_3}+A_3\right|\geq 1.$$

Since

$$\left|A_1\delta^{m_1-m_3}+A_2\delta^{m_1-m_3}+A_3\right|\leq 3X,$$

which implies $|A_1\gamma^{m_1-m_3}+A_2\gamma^{m_2-m_3}+A_3|\geq\frac{1}{3X}$. As a result, we have

$$\left|\frac{A_1}{\gamma^{m_1}}+\frac{A_2}{\gamma^{m_2}}+\frac{A_3}{\gamma^{m_3}}\right|\leq\frac{3X}{\gamma^{m_3}}.$$

Hence,

$$\left|\gamma^{m_3}\left(A_1\gamma^{m_1-m_3}+A_2\gamma^{m_2-m_3}+A_3\right)-\left(\frac{A_1}{\gamma^{m_1}}+\frac{A_2}{\gamma^{m_2}}+\frac{A_3}{\gamma^{m_3}}\right)\right|\geq\frac{\gamma^{m_3}}{3X}-\frac{3X}{\gamma^{m_3}}.$$

If $\frac{\gamma^{m_3}}{3X}-\frac{3X}{\gamma^{m_3}}\leq\frac{\gamma^{m_3}}{6X}$, then $\gamma^{2m_3}\leq 18X^2$, so $\gamma^{m_3}<5X$. Hence,

$$m_6\leq m_5\leq m_4\leq m_3\leq\frac{\log(5X)}{\log\psi}. \tag{3.8}$$

Using (3.1) and (3.6), we have

$$m_2\leq\frac{\log(320X^3)}{\log\psi} \text{ and } m_1\leq\frac{\log(3520X^4)}{\log\psi}. \tag{3.9}$$

Now, assume $\frac{\gamma^{m_3}}{6X} \leq \frac{\gamma^{m_3}}{3X} - \frac{3X}{\gamma^{m_3}} \leq 6X\gamma^{m_4}$, then $\gamma^{m_3-m_4} \leq 36X^2$. Hence,

$$m_3 - m_4 \leq 2\frac{\log(6X)}{\log\psi}. \tag{3.10}$$

If we rewrite (2.8), then we have

$$\begin{aligned}\left|\gamma^{m_4}\left(A_1\frac{\gamma^{m_1}}{\gamma^{m_4}} + A_2\frac{\gamma^{m_2}}{\gamma^{m_4}} + A_3\frac{\gamma^{m_3}}{\gamma^{m_4}} + A_4\right) - \left(\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}} + \frac{A_3}{\gamma^{m_3}} + \frac{A_4}{\gamma^{m_4}}\right)\right| \\ = \left|-A_5(\gamma^{m_5} - \delta^{m_5}) - A_6(\gamma^{m_6} - \delta^{m_6})\right| \\ \leq 4X\gamma^{m_5}. \qquad (3.11)\end{aligned}$$

Assume $A_1\gamma^{m_1-m_4} + A_2\gamma^{m_2-m_4} + A_3\gamma^{m_3-m_4} = 0$. Let $i = m_3 - m_4, j = m_2 - m_4, k = m_1 - m_4$. Then

$$k \leq \frac{\log(25,344X^5)}{\log\psi},\ j \leq \frac{\log(2304X^4)}{\log\psi} \text{ and } i \leq 2\frac{\log(6X)}{\log\psi}$$

are bounded. Therefore one can find all polynomials $A_1X^k + A_2X^j + A_3X^i + A_4$ and took that polynomial which has a root γ, where γ is a quadratic unit of norm 1. Since δ is also a root of the same polynomial for these balancing-like sequences, the left-hand side of (3.11), for any m_4, is zero. This proves that $m_5 = m_6 = 0$ as well. Next, we have

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (m_4 + k, m_4 + j, m_4 + i, m_4, 0, 0)$$

is a parametric family of solutions. From now on assume that $A_1\gamma^{m_1-m_4} + A_2\gamma^{m_2-m_4} + A_3\gamma^{m_3-m_4} + A_4 \neq 0$. Thus,

$$\begin{aligned}&\left|A_1\gamma^{m_1-m_4} + A_2\gamma^{m_2-m_4} + A_3\gamma^{m_3-m_4} + A_4\right| \\ &\times \left|A_1\delta^{n_m-m_4} + A_2\delta^{m_2-m_4} + A_3\delta^{m_3-m_4} + A_4\right| \geq 1.\end{aligned}$$

Since $|A_1\delta^{m_1-m_4} + A_2\delta^{m_2-m_4} + A_3\delta^{m_3-m_4} + A_4| \leq 4X$, which implies

$$\left|A_1\gamma^{m_1-m_4} + A_2\gamma^{m_2-m_4} + A_3\gamma^{m_3-m_4} + A_4\right| \geq \frac{1}{4X}.$$

As a result, we have

$$\left|\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}} + \frac{A_3}{\gamma^{m_3}} + \frac{A_4}{\gamma^{m_4}}\right| \leq \frac{4X}{\gamma^{m_4}}.$$

Hence,

$$\begin{aligned}\left|\gamma^{m_4}\left(A_1\frac{\gamma^{m_1}}{\gamma^{m_4}} + A_2\frac{\gamma^{m_2}}{\gamma^{m_4}} + A_3\frac{\gamma^{m_3}}{\gamma^{m_4}} + A_4\right) - \left(\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}} + \frac{A_3}{\gamma^{n_3}} + \frac{A_4}{\gamma^{n_4}}\right)\right| \\ \geq \frac{\gamma^{m_4}}{4X} - \frac{4X}{\gamma^{m_4}}.\end{aligned}$$

If $\frac{\gamma^{m_4}}{4X} - \frac{4X}{\gamma^{m_4}} \leq \frac{\gamma^{m_4}}{8X}$, then $\gamma^{2m_4} \leq 32X^2$, so $\gamma^{m_4} < 6X$. Hence,

$$m_6 \leq m_5 \leq m_4 \leq \frac{\log(6X)}{\log\psi}. \tag{3.12}$$

Using (3.1), (3.6) and (3.10), we have

$$m_3 \le \frac{3\log(6X)}{\log\psi},\ m_2 \le \frac{\log(13824X^5)}{\log\psi} \text{ and } m_1 \le \frac{\log(152064X^6)}{\log\psi}. \tag{3.13}$$

Now, assume $\frac{\gamma^{m_4}}{8X} \le \frac{\gamma^{m_4}}{4X} - \frac{4X}{\gamma^{m_4}} \le 4X\gamma^{m_5}$, then $\gamma^{m_4-m_5} \le 32X^2$. Hence,

$$m_4 - m_5 \le \frac{\log(32X^2)}{\log\psi}. \tag{3.14}$$

Again, rewriting (2.8), we have

$$\begin{aligned} &\Big|\gamma^{m_5}\left(A_1\gamma^{m_1-m_5} + A_2\gamma^{m_2-m_5} + A_3\gamma^{m_3-m_5} + A_4\gamma^{m_4-m_5} + A_5\right) \\ &\quad - \left(\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}} + \frac{A_3}{\gamma^{m_3}} + \frac{A_4}{\gamma^{m_4}} + \frac{A_5}{\gamma^{m_5}}\right)\Big| \\ &= |-A_6(\gamma^{m_6} - \delta^{m_6})| \\ &\le 2X\gamma^{m_6}. \end{aligned} \tag{3.15}$$

Assume $A_1\gamma^{m_1-m_5} + A_2\gamma^{m_2-m_5} + A_3\gamma^{m_3-m_5} + A_4\gamma^{m_4-m_5} + A_5 = 0$. Let $i = m_4 - m_5, j = m_3 - m_5, k = m_2 - m_5, l = m_1 - m_5$. Then

$$l \le \frac{\log(811008X^7)}{\log\psi}, k \le \frac{\log(73728X^6)}{\log\psi}, j \le \frac{\log(1152X^4)}{\log\psi} \text{ and } i \le \frac{\log(32X^2)}{\log\psi}$$

are bounded. Therefore one can find all polynomials $A_1X^l + A_2X^k + A_3X^j + A_4X^i + A_5$ and took that polynomial which has a root γ, where γ is a quadratic unit of norm 1. Since δ is also a root of the same polynomial for these balancing-like sequences, the left-hand side of (3.15), for any m_5, is zero. This proves that $m_6 = 0$ as well. Next, we have

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (m_5 + l, m_5 + k, m_5 + j, m_5 + i, m_5, 0)$$

is a parametric family of solutions. From now on, assume that

$$A_1\gamma^{m_1-m_5} + A_2\gamma^{m_2-m_5} + A_3\gamma^{m_3-m_5} + A_4\gamma^{m_4-m_5} + A_5 \ne 0.$$

Thus,

$$\begin{aligned} &\left|A_1\gamma^{m_1-m_5} + A_2\gamma^{m_2-m_5} + A_3\gamma^{m_3-m_5} + A_4\gamma^{m_4-m_5} + A_5\right| \\ &\times \left|A_1\delta^{m_1-m_5} + A_2\delta^{m_2-m_5} + A_3\delta^{m_3-m_5} + A_4\delta^{m_4-m_5} + A_5\right| \ge 1. \end{aligned}$$

Since $|A_1\delta^{m_1-m_5} + A_2\delta^{m_2-m_5} + A_3\delta^{m_3-m_5} + A_4\delta^{m_4-m_5} + A_5| \le 5X$, which implies

$$\left|A_1\gamma^{m_1-m_5} + A_2\gamma^{m_2-m_5} + A_3\gamma^{m_3-m_5} + A_4\gamma^{m_4-m_5} + A_5\right| \ge \frac{1}{5X}.$$

As a result, we have

$$\left|\frac{A_1}{\gamma^{m_1}} + \frac{A_2}{\gamma^{m_2}} + \frac{A_3}{\gamma^{m_3}} + \frac{A_4}{\gamma^{m_4}} + \frac{A_5}{\gamma^{m_5}}\right| \le \frac{5X}{\gamma^{m_5}}.$$

Hence, the left-hand side of (3.15) is at least as large as

$$\frac{\gamma^{m_5}}{5X} - \frac{5X}{\gamma^{m_5}}.$$

If $\frac{\gamma^{m_5}}{5X} - \frac{5X}{\gamma^{m_5}} \leq \frac{\gamma^{m_5}}{10X}$, then $\gamma^{2m_5} \leq 50X^2$, so $\gamma^{m_5} < 8X$. Hence,

$$m_6 \leq m_5 \leq \frac{\log(8X)}{\log\psi}. \tag{3.16}$$

Using (3.1), (3.6), (3.10) and (3.14), we have

$$m_4 \leq \frac{\log(256X^3)}{\log\psi}, m_3 \leq \frac{\log(9216X^5)}{\log\psi}, m_2 \leq \frac{\log(589824X^7)}{\log\psi}$$

$$\text{and } m_1 \leq \frac{\log(6488064X^8)}{\log\psi}. \tag{3.17}$$

Now, assume $\frac{\gamma^{m_5}}{10X} \leq \frac{\gamma^{m_5}}{5X} - \frac{5X}{\gamma^{m_5}} \leq 2X\gamma^{m_6}$, then $\gamma^{m_5-m_6} \leq 20X^2$. Hence,

$$m_5 - m_6 \leq \frac{\log(20X^2)}{\log\psi}. \tag{3.18}$$

Again we rewrite (2.8) to have

$$\begin{aligned} &\left|\gamma^{m_6}\left(A_1\frac{\gamma^{m_1}}{\gamma^{m_6}} + A_2\frac{\gamma^{m_2}}{\gamma^{m_6}} + A_3\frac{\gamma^{m_3}}{\gamma^{m_6}} + A_4\frac{\gamma^{m_4}}{\gamma^{m_6}} + A_5\frac{\gamma^{m_5}}{\gamma^{m_6}} + A_6\right)\right| \\ &= \left|\delta^{m_6}\left(A_1\frac{\delta^{m_1}}{\delta^{m_6}} + A_2\frac{\delta^{m_2}}{\delta^{m_6}} + A_3\frac{\delta^{m_3}}{\delta^{m_6}} + A_4\frac{\delta^{m_4}}{\delta^{m_6}} + A_5\frac{\delta^{m_5}}{\delta^{m_6}} + A_6\right)\right|. \end{aligned} \tag{3.19}$$

Assume that the left-hand side of (3.19) is zero. Let $i = m_5 - m_6, j = m_4 - m_6, k = m_3 - m_6, l = m_2 - m_6, m = m_1 - m_6$. Then

$$m \leq \frac{\log(16220160X^9)}{\log\psi},\ l \leq \frac{\log(1474560X^8)}{\log\psi},\ k \leq \frac{\log(23040X^6)}{\log\psi},$$
$$j \leq \frac{\log(640X^4)}{\log\psi} \text{ and } i \leq \frac{\log(20X^2)}{\log\psi}$$

are bounded. Therefore one can find all polynomials $A_1X^m + A_2X^l + A_3X^k + A_4X^j + A_5X^i + A_6$ and took that polynomial which has a root γ, where γ is a quadratic unit of norm 1. For such, (3.19) holds for all m_6. Hence, we have the parametric family of solutions

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (m_6 + m, m_6 + l, m_6 + k, m_6 + j, m_6 + i, m_6).$$

From now on we assume that the left-hand side of (3.19) is nonzero. Thus,

$$\begin{aligned} &\left|A_1\frac{\gamma^{m_1}}{\gamma^{m_6}} + A_2\frac{\gamma^{m_2}}{\gamma^{m_6}} + A_3\frac{\gamma^{m_3}}{\gamma^{m_6}} + A_4\frac{\gamma^{m_4}}{\gamma^{m_6}} + A_5\frac{\gamma^{m_5}}{\gamma^{m_6}} + A_6\right| \\ &\times \left|A_1\frac{\delta^{m_1}}{\delta^{m_6}} + A_2\frac{\delta^{m_2}}{\delta^{m_6}} + A_3\frac{\delta^{m_3}}{\delta^{m_6}} + A_4\frac{\delta^{m_4}}{\delta^{m_6}} + A_5\frac{\delta^{m_5}}{\delta^{m_6}} + A_6\right| \geq 1. \end{aligned}$$

The second factor on the left-hand side of the above equation is $\leq 6X$ and hence

$$\left|A_1\frac{\gamma^{m_1}}{\gamma^{m_6}}+A_2\frac{\gamma^{m_2}}{\gamma^{m_6}}+A_3\frac{\gamma^{m_3}}{\gamma^{m_6}}+A_4\frac{\gamma^{m_4}}{\gamma^{m_6}}+A_5\frac{\gamma^{m_5}}{\gamma^{m_6}}+A_6\right|\geq\frac{1}{6X}.$$

Thus, (3.19) becomes

$$\frac{\gamma^{m_6}}{6X}\leq 6\delta^{m_6}=\frac{6X}{\gamma^{m_6}},$$

which gives

$$m_6\leq\frac{\log(6X)}{\log\psi}. \tag{3.20}$$

Using (3.1), (3.6), (3.10), (3.14) and (3.18), we have

$$\begin{aligned} m_6&\leq\frac{\log(6X)}{\log\psi},\\ m_5&\leq\frac{\log(120X^3)}{\log\psi},\\ m_4&\leq\frac{\log(3840X^5)}{\log\psi},\\ m_3&\leq\frac{\log(138240X^7)}{\log\psi},\\ m_2&\leq\frac{\log(8847360X^9)}{\log\psi},\\ m_1&\leq\frac{\log(97320960X^{10})}{\log\psi}. \end{aligned} \tag{3.21}$$

Note that (3.21) contains (3.16), (3.17), (3.12), (3.13), (3.8), (3.9), (3.4) and (3.5). Keeping in mind that we might need to substitute $2X$ instead of X and this completes the proof of the Theorem 1.3.

4. Numerical example

Take $A_1, A_2, A_3, A_4, A_5, A_6 \in \{0, \pm1\}$. Hence $X = 1$, therefore $A \leq 308$. Thus, Theorem 1.3 says that the sporadic solutions are of the form

$$x_{m_1}\pm A_2x_{m_2}\pm A_3x_{m_3}\pm A_4x_{m_4}\pm A_5x_{m_5}\pm A_6x_{m_6}=0,$$

where $A_2, A_3, A_4, A_5, A_6 \in \{0, \pm1\}$, $m_1 > m_2 \geq m_3 \geq m_4 \geq m_5 \geq m_6 \geq 0$. Here $m_6 < 3, m_5 < 8, m_4 < 13, m_3 < 18, m_2 < 24$ and $m_1 > m_2$. Now for $r \in [3, 308], m_6 \in [0, 3], m_5 \in [m_6, 8], m_4 \in [m_5, 13], m_3 \in [m_4, 24], m_2 \in [m_3, 24], \alpha_1 \in \{0, 1\}$ and $\alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \{0, \pm1\}$, we search the equation

$$x_{m_1}=\mid\alpha_2x_{m_2}+\alpha_3x_{m_3}+\alpha_4x_{m_4}+\alpha_5x_{m_5}+\alpha_6x_{m_6}\mid,$$

holds for some $m_1 > m_2$. Using *Mathematica* software, we found 127 solutions. Out of them, 114 solutions correspond to $A = 3$, 12 solutions correspond to $A = 4$ and only one solution $x_1 + x_1 + x_1 + x_1 + x_1 = x_2$ corresponds to $A = 5$.

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