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ON NEXT-TO-MINIMUM SIZE BLOCKING SETS OF EXTERNAL LINES TO A NONDEGENERATE QUADRIC IN PG(3, q)

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Abstract. Consider a hyperbolic or an elliptic quadric $Q^\epsilon(3,q)$, $\epsilon \in \{+,-\}$, in $\mathrm{PG}(3,q)$ and let $\mathcal{E}^\epsilon$ denote the set of all lines of $\mathrm{PG}(3,q)$ that are external with respect to $Q^\epsilon(3,q)$. If π is a (tangent or secant) plane of $\mathrm{PG}(3,q)$, then $\pi \setminus Q^\epsilon(3,q)$ is an $\mathcal{E}^\epsilon$-blocking set which is minimal, except when $(q,\epsilon) = (2,+)$ and π is a secant plane. In this way, we obtain two families of (minimal) blocking sets of sizes m_1^ϵ and m_2^ϵ, with $(m_1^+, m_2^+) = (q^2-q, q^2)$ and $(m_1^-, m_2^-) = (q^2, q^2+q)$, where those of size m_1^ϵ are also the minimum size $\mathcal{E}^\epsilon$-blocking sets. Motivated by the search for new (families of) minimal $\mathcal{E}^\epsilon$-blocking sets with sizes in the open interval $]m_1^\epsilon, m_2^\epsilon[$, we determine here all $\mathcal{E}^\epsilon$-blocking sets of size $m_1^\epsilon + 1$ in $\mathrm{PG}(3,q)$ in the case $q \in \{2,3\}$.

1. Introduction

Let q be a prime power and $\mathrm{PG}(3,q)$ be the three–dimensional projective space over the finite field $\mathbb{F}_q$ of order q. The unique line of $\mathrm{PG}(3,q)$ through two distinct points x and y is denoted by xy. For a given nonempty set $\mathcal{L}$ of lines of $\mathrm{PG}(3,q)$, a set X of points of $\mathrm{PG}(3,q)$ is called an *$\mathcal{L}$-blocking set* if each line in $\mathcal{L}$ contains at least one point of X. An $\mathcal{L}$-blocking set X is said to be *minimal* if no proper subset of X is an $\mathcal{L}$-blocking set in

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PG$(3, q)$. Clearly, every minimum size $\mathcal{L}$-blocking set in PG$(3, q)$ is also a minimal $\mathcal{L}$-blocking set. The first step in the study of blocking sets has been to determine the smallest cardinality of a blocking set and to characterize, if possible, all blocking sets of that cardinality. If $\mathcal{L}$ is the set of all lines of PG$(3, q)$ and X is an $\mathcal{L}$-blocking set in PG$(3, q)$, then a classical result of Bose and Burton [4, Theorem 1] gives that $|X| \geq q^2 + q + 1$, with equality if and only if X is the point set of a plane of PG$(3, q)$.

1.1. **Irreducible conics in** PG$(2, q)$**.** Let $\mathcal{C}$ be an irreducible conic in PG$(2, q)$. We refer to [5, Chapter 7] or [11, Chapter III] for the following basic properties of points and lines of PG$(2, q)$ with respect to $\mathcal{C}$. The conic $\mathcal{C}$ contains $q+1$ points and meets every line of PG$(2, q)$ in at most two points. A line of PG$(2, q)$ is called *external* (respectively, *tangent, secant*) with respect to $\mathcal{C}$ if it meets $\mathcal{C}$ in 0 (respectively, 1, 2) points. Every point of $\mathcal{C}$ is contained in a unique tangent line, giving $q+1$ tangent lines to $\mathcal{C}$. There are $q(q+1)/2$ secant lines and so $(q^2+q+1)-(q+1)-q(q+1)/2 = q(q-1)/2$ external lines to $\mathcal{C}$. Every point of $\mathcal{C}$ is contained in q secant lines.

Suppose that q is even. Then all the $q+1$ tangent lines meet in a common point and this common point of intersection is called the *nucleus* of $\mathcal{C}$. Every point of PG$(2, q) \setminus \mathcal{C}$ different from the nucleus lies on $q/2$ secant lines, $q/2$ external lines and a unique tangent line.

Suppose that q is odd. Then every point of PG$(2, q) \setminus \mathcal{C}$ lies on 0 or 2 tangent lines. Such a point is called *interior* to $\mathcal{C}$ in the first case and *exterior* to $\mathcal{C}$ in the latter. There are $q(q-1)/2$ interior points and $q(q+1)/2$ exterior points with respect to $\mathcal{C}$. Every interior point lies on $(q+1)/2$ external lines and $(q+1)/2$ secant lines. Every exterior point lies on $(q-1)/2$ external lines and $(q-1)/2$ secant lines. Every external line contains $(q+1)/2$ interior points and $(q+1)/2$ exterior points. Every secant line contains $(q-1)/2$ interior points and $(q-1)/2$ exterior points.

The following result on the size of blocking sets of external lines in PG$(2, q)$ was proved by Aguglia and Korchmáros in [1, Theorem 1.1] for q odd and by Giulietti in [9, Theorem 1.1] for q even (they also characterized the equality case).

Proposition 1.1 ([1, 9])**.** *Let A be a blocking set in* PG$(2, q)$ *of the external lines with respect to an irreducible conic. Then $|A| \geq q - 1$.*

1.2. **Nondegenerate quadrics in** PG$(3, q)$**.** There are two types of nondegenerate quadrics in PG$(3, q)$: (1) Elliptic quadrics $Q^-(3, q)$ which are of Witt index one, and (2) Hyperbolic quadrics $Q^+(3, q)$ which are of Witt index two. We refer to [10] for the basic properties of points, lines and planes of PG$(3, q)$ with respect to a nondegenerate quadric in it.

Let $Q^-(3, q)$ be an elliptic quadric in PG$(3, q)$. Then $Q^-(3, q)$ contains q^2+1 points and it meets every line in at most two points. A line of PG$(3, q)$ is called *external*, *tangent* or *secant* with respect to $Q^-(3, q)$ according as it meets $Q^-(3, q)$ in $0, 1$ or 2 points. Every point of $Q^-(3, q)$ is contained in

$q+1$ tangent lines, this gives $(q+1)(q^2+1)$ tangent lines to $Q^-(3,q)$. We also have $q^2(q^2+1)/2$ secant lines and then $(q^2+1)(q^2+q+1) - q^2(q^2+1)/2 - (q+1)(q^2+1) = q^2(q^2+1)/2$ external lines to $Q^-(3,q)$. Every point of $Q^-(3,q)$ is contained in q^2 secant lines. Every point of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ is contained in $q+1$ tangent lines, $q(q-1)/2$ secant lines and $q(q+1)/2$ external lines.

Let $Q^+(3,q)$ be a hyperbolic quadric in $\mathrm{PG}(3,q)$. Then $Q^+(3,q)$ contains $(q+1)^2$ points. Every line of $\mathrm{PG}(3,q)$ meets $Q^+(3,q)$ in $0,1,2$ or $q+1$ points. A line of $\mathrm{PG}(3,q)$ is called *external* or *secant* with respect to $Q^+(3,q)$ according as it meets $Q^-(3,q)$ in 0 or 2 points. A line of $\mathrm{PG}(3,q)$ is called an *outer tangent* or an *inner tangent* with respect to $Q^+(3,q)$ according as it meets $Q^-(3,q)$ in 1 or $q+1$ points. Every point of $Q^+(3,q)$ is contained in 2 inner tangents, $q-1$ outer tangents and q^2 secant lines. Every point of $\mathrm{PG}(3,q) \setminus Q^+(3,q)$ is contained in $q+1$ outer tangents, $q(q+1)/2$ secant lines and $q(q-1)/2$ external lines. There are $2(q+1)$ inner tangents, $(q-1)(q+1)^2$ outer tangents, $q^2(q+1)^2/2$ secant lines and $q^2(q-1)^2/2$ external lines to $Q^+(3,q)$.

With the quadric $Q^\epsilon(3,q)$, $\epsilon \in \{-,+\}$, there is naturally associated a polarity τ which is symplectic if q is even, and orthogonal if q is odd. For a point x of $\mathrm{PG}(3,q)$, the plane x^τ is called a *tangent* or *secant* plane according as x is a point of $Q^\epsilon(3,q)$ or not. For every point x of $\mathrm{PG}(3,q) \setminus Q^\epsilon(3,q)$, the secant plane x^τ intersects $Q^\epsilon(3,q)$ in an irreducible conic. For every point x of $Q^\epsilon(3,q)$, the tangent plane x^τ intersects $Q^\epsilon(3,q)$ at the point x if $\epsilon = -$, and in the union of two inner tangents through x if $\epsilon = +$. In both cases, the $q+1$ tangent lines through x are precisely the lines through x contained in x^τ.

Suppose that q is odd. Then, for every point x of $\mathrm{PG}(3,q) \setminus Q^\epsilon(3,q)$, the secant plane x^τ does not contain the point x and the tangent lines through x are precisely the $q+1$ lines through x meeting the conic $x^\tau \cap Q^\epsilon(3,q)$.

1.3. **Blocking sets of external lines in** $\mathrm{PG}(3,q)$.

We shall denote by $\mathcal{E}^-$ and $\mathcal{E}^+$ the set of all external lines of $\mathrm{PG}(3,q)$ with respect to $Q^-(3,q)$ and $Q^+(3,q)$, respectively. For $\epsilon \in \{-,+\}$, two $\mathcal{E}^\epsilon$-blocking sets X_1 and X_2 in $\mathrm{PG}(3,q)$ are said to be *isomorphic* if there is an automorphism of $\mathrm{PG}(3,q)$ stabilizing $Q^\epsilon(3,q)$ and mapping X_1 to X_2.

In [3, Theorem 3.5], Biondi et al. studied $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,q)$ and characterized such blocking sets of minimum size for $q \geq 9$ (their proof also works for $q \in \{2,4,8\}$). Recently, an alternate proof characterizing the minimum size $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,q)$ was given in [6, Theorem 1.7] by the authors which works for all q. More precisely, the following result holds for $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,q)$:

Proposition 1.2 ([3, 6]). *Let X be an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,q)$. Then $|X| \geq q^2$, and equality holds if and only if $X = \pi \setminus Q^-(3,q)$ for some secant plane π of $\mathrm{PG}(3,q)$ with respect to $Q^-(3,q)$.*

If π is a tangent plane of PG$(3, q)$ with respect to $Q^-(3, q)$, then $\pi \setminus Q^-(3, q)$ is obviously also an $\mathcal{E}^-$-blocking set. A straightforward counting shows that each point of $\pi \setminus Q^-(3, q)$ is contained in an external line that does not lie in π, implying that also this $\mathcal{E}^-$-blocking set is minimal. Its size is equal to $q^2 + q$. It can be of interest to search for new (families of) minimal $\mathcal{E}^-$-blocking sets whose sizes are relatively small, say in the open interval $]q^2, q^2 + q[$. In this paper, we classify all minimal $\mathcal{E}^-$-blocking sets if $q = 2$ and all minimal $\mathcal{E}^-$-blocking sets of size $q^2 + 1 = 10$ if $q = 3$. We have not been able so far to describe an infinite family that contains these examples.

The minimum size $\mathcal{E}^+$-blocking sets in PG$(3, q)$ were characterized in [2, Theorem 1.1] for even $q \geq 8$ and in [3, Theorem 2.4] for odd $q \geq 9$. Alternate proofs characterizing such blocking sets are given in [7, Section 2] for all odd q and in [12, Section 3] for all even q. More precisely, the following result holds for $\mathcal{E}^+$-blocking sets in PG$(3, q)$:

Proposition 1.3 ([2, 3, 7, 12])**.** *Let X be an $\mathcal{E}^+$-blocking set in* PG$(3, q)$*. Then $|X| \geq q^2 - q$, and equality holds if and only if $X = \pi \setminus Q^+(3, q)$ for some tangent plane π of* PG$(3, q)$ *with respect to $Q^+(3, q)$.*

If π is a secant plane of PG$(3, q)$ with respect to $Q^+(3, q)$, then $\pi \setminus Q^+(3, q)$ is obviously also an $\mathcal{E}^+$-blocking set. It is straightforward to verify that each point of $\pi \setminus Q^+(3, q)$ is contained in an external line that is not in π if and only if $q \neq 2$. So, for $q > 2$, this $\mathcal{E}^+$-blocking set is minimal. Its size is equal to q^2. It can be of interest to search for new (families of) minimal $\mathcal{E}^+$-blocking sets whose sizes are relatively small, say in the open interval $]q^2 - q, q^2[$. In this paper, we classify all minimal $\mathcal{E}^+$-blocking sets if $q = 2$ and all minimal $\mathcal{E}^+$-blocking sets of size $q^2 - q + 1 = 7$ if $q = 3$. We have not been able so far to describe an infinite family that contains these examples.

1.4. **Description of the main results.** Suppose first that $q = 2$ and consider a hyperbolic quadric $Q^+(3, 2)$ in PG$(3, 2)$. If π is a tangent plane of PG$(3, 2)$ with respect to $Q^+(3, 2)$, then by Proposition 1.3 $\pi \setminus Q^+(3, 2)$ is a minimal $\mathcal{E}^+$-blocking set in PG$(3, 2)$ of size 2. Such an $\mathcal{E}^+$-blocking set is also of the form $L \setminus Q^+(3, 2)$ for some outer tangent L. We will prove the following.

Theorem 1.4. *Every minimal $\mathcal{E}^+$-blocking set in* PG$(3, 2)$ *is of the form $L \setminus Q^+(3, 2)$ for some outer tangent L.*

Consider now the elliptic quadric $Q^-(3, 2)$ in PG$(3, 2)$. Let V be the 4-dimensional vector space over $\mathbb{F}_2$ for which PG$(3, 2)$ is the associated projective space. The elliptic quadrics in PG$(3, 2)$ are precisely the frames of PG$(3, 2)$ implying that we can take an ordered basis $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ in V with respect to which $Q^-(3, 2)$ consists of the points with coordinates $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ and $(1, 1, 1, 1)$. Let Ω denote the set of all ω's of the form $(\{i, j\}, k, l)$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and $i < j$. For every such $\omega \in \Omega$, we define

$$B_\omega := \{\langle \bar{e}_i + \bar{e}_j \rangle, \langle \bar{e}_i + \bar{e}_j + \bar{e}_k \rangle, \langle \bar{e}_i + \bar{e}_j + \bar{e}_l \rangle, \langle \bar{e}_i + \bar{e}_k \rangle, \langle \bar{e}_j + \bar{e}_l \rangle\}.$$

The set $\mathcal{B} := \{B_\omega \,|\, \omega \in \Omega\}$ contains 12 elements which are all equivalent under the stabilizer of $Q^-(3,2)$ inside $PGL(4,2)$, which is a group isomorphic to S_5. We will prove the following.

Theorem 1.5. *Each element of $\mathcal{B}$ is a minimal $\mathcal{E}^-$-blocking set of size 5.*

Theorem 1.6. *Up to isomorphisms, there are three minimal $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,2)$. Each such blocking set is either of the form $\pi \setminus Q^-(3,2)$ for some secant plane π, of the form $\pi \setminus Q^-(3,2)$ for some tangent plane π or belongs to $\mathcal{B}$.*

Next, we describe our obtained results for $q = 3$. For a given secant plane π of $\mathrm{PG}(3,3)$ with respect to the quadric $Q^\epsilon(3,3)$ with $\epsilon \in \{-,+\}$, we denote by E_π the set of six points of π that are exterior with respect to the irreducible conic $\pi \cap Q^\epsilon(3,3)$ in π. We will prove the following results.

Theorem 1.7. *Let $Q^-(3,3)$ be the elliptic quadric in $\mathrm{PG}(3,3)$ that has equation $X_1X_2 + X_3^2 + X_4^2 = 0$ with respect to a certain reference system in $\mathrm{PG}(3,3)$ and let π be the secant plane of $\mathrm{PG}(3,3)$ with equation $X_4 = 0$. Then*

$$Y := E_\pi \cup \{\langle(1,0,0,1)\rangle, \langle(0,-1,0,1)\rangle, \langle(1,-1,1,1)\rangle, \langle(1,-1,-1,1)\rangle\}$$

is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10.

Theorem 1.8. *Let B be an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10. Then B is one of the following:*

(1) *$B = (\pi \setminus Q^-(3,3)) \cup \{x\}$, where π is a secant plane of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ and x is a point of $\mathrm{PG}(3,3)$ not belonging to $\pi \setminus Q^-(3,3)$.*

(2) *B is isomorphic to the $\mathcal{E}^-$-blocking set Y described in Theorem 1.7.*

Theorem 1.9. *Let w be a point of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ and π be the secant plane w^τ, where τ is the orthogonal polarity associated with $Q^+(3,3)$. Then $B := E_\pi \cup \{w\}$ is a minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$ of size 7.*

Theorem 1.10. *Let B be an $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$ of size 7. Then B is one of the following:*

(1) *$B = (\pi \setminus Q^+(3,3)) \cup \{x\}$, where π is a tangent plane of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$ and x is a point of $\mathrm{PG}(3,3)$ not belonging to $\pi \setminus Q^+(3,3)$.*

(2) *B is as described in Theorem 1.9.*

2. Proofs of Theorems 1.4, 1.5 and 1.6

2.1. **Proof of Theorem 1.4.** Suppose $q = 2$ and X is a minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,2)$. Then $|X| \geq 2$ with equality if and only if $X = L \setminus$

$Q^+(3,2)$ for some outer tangent L (see Proposition 1.3). Suppose therefore that $|X| \geq 3$. As X is minimal and $X \setminus Q^+(3,2)$ is an $\mathcal{E}^+$-blocking set, we know that $X \cap Q^+(3,2) = \emptyset$ and no two distinct points of X are on the same outer tangent.

Let x_1, x_2 and x_3 be three mutually distinct points of X. Suppose they are on an (external) line L of PG$(3,2)$. Let π be a (secant) plane through L and let y be the nucleus of the conic $\pi \cap Q^+(3,2)$. The unique external line through y is not contained in π and contains a point $x_3' \in X$. Upon replacing x_3 by x_3', we may thus assume that x_1, x_2 and x_3 are three points of X not on the same line of PG$(3,2)$. But then x_1x_2, x_1x_3 and x_2x_3 are three distinct external lines of the plane $\langle x_1, x_2, x_3\rangle$. This is impossible as a plane can only contain 0 or 1 external line depending on whether it is a tangent or secant plane.

2.2. **Proof of Theorem 1.5.** Suppose $q = 2$. We first give here a construction for minimal $\mathcal{E}^-$-blocking sets in PG$(3,2)$ of size 5 and prove subsequently that each element of $\mathcal{B}$ can be obtained via this construction.

Let π be a secant plane of PG$(3,2)$ with respect to $Q^-(3,2)$ and denote by k the nucleus of the irreducible conic $\mathcal{C}_\pi = \pi \cap Q^-(3,2)$. Let x be an arbitrary point of $\pi \setminus (\mathcal{C}_\pi \cup \{k\})$ and denote the two other points of $\pi \setminus (\mathcal{C}_\pi \cup \{k\})$ by y_1 and y_2. Through x, there are two external lines L_1 and L_2 not contained in π. The plane $\langle L_1, L_2\rangle$ meets π in a third line L_3 through x. Since L_1 and L_2 are external lines, $\langle L_1, L_2\rangle$ must be a tangent plane and L_3 a tangent line necessarily equal to kx. Let $x_1 \in L_1 \setminus \{x\}$ and $x_2 \in L_2 \setminus \{x\}$ such that the line x_1x_2 contains the tangency point x_3 of L_3 (so $L_3 = \{k, x, x_3\}$). By construction, $X := \{k, y_1, y_2, x_1, x_2\}$ is an $\mathcal{E}^-$-blocking set in PG$(3,2)$ of size 5. We now show that it is minimal. As L_i is an external line meeting X precisely in x_i, $X \setminus \{x_i\}$ is not an $\mathcal{E}^-$-blocking set for every $i \in \{1,2\}$. As $\langle L_1, L_2\rangle$ is a tangent plane with tangency point x_3, the lines x_1k and x_2k are external to $Q^-(3,2)$. As the third external line through k is not contained in π, we see that $X \setminus \{k\}$ is not an $\mathcal{E}^-$-blocking set. Suppose now that $X \setminus \{y_i\}$ is an $\mathcal{E}^-$-blocking set for some $i \in \{1,2\}$. Then y_ix_1 and y_ix_2 must be the two external lines through y_i not contained in π. As above, we then know that the plane $\langle y_ix_1, y_ix_2\rangle$ meets π in the line ky_i, in particular the line x_1x_2 must meet ky_i. This is obviously not the case here as the point $x_3 \in kx$ does not lie on ky_i. So, X must be a minimal $\mathcal{E}^-$-blocking set.

Lemma 2.1. *Each $B \in \mathcal{B}$ is an $\mathcal{E}^-$-blocking set in* PG$(3,2)$ *that can be obtained via the above construction.*

Proof. Let π be the secant plane through the points $(1,0,0,0)$, $(0,1,0,0)$ and $(0,0,1,0)$ of $Q^-(3,2)$. The nucleus k of $\mathcal{C}_\pi = \pi \cap Q^-(3,2)$ is equal to $(1,1,1,0)$ and we denote the three points of $\pi \setminus (\mathcal{C}_\pi \cup \{k\})$ by $x = (0,1,1,0)$, $y_1 = (1,1,0,0)$ and $y_2 = (1,0,1,0)$. The tangency point x_3 on the line $L_3 = kx$ is then equal to $(1,0,0,0)$. If $x_1 = (0,1,0,1)$ and $x_2 = (1,1,0,1)$, then $x_3 \in x_1x_2$ and $L_1 = xx_1$, $L_2 = xx_2$ are the two external lines through

x not contained in π. We thus see that B_ω with $\omega = (\{1,2\},3,4)$, which is equal to the $\mathcal{E}^-$-blocking set $\{k, y_1, y_2, x_1, x_2\}$, can be obtained as in the above construction. The claim then follows from the fact that any two elements of $\mathcal{B}$ are isomorphic. □

2.3. **Proof of Theorem 1.6.** Suppose again that $q = 2$. We already know that $\pi \setminus Q^-(3,2)$ is a minimal $\mathcal{E}^-$-blocking set in PG(3, 2) for every plane π. Let B be a minimal $\mathcal{E}^-$-blocking set in PG(3, 2) that is not of the form $\pi \setminus Q^-(3,2)$ for some plane π. As B is minimal and $B \setminus Q^-(3,2)$ is a minimal $\mathcal{E}^-$-blocking set, we know that $B \cap Q^-(3,2) = \emptyset$ and B does not contain (nor is contained in) a set of the form $\pi \setminus Q^-(3,2)$ with π a plane.

Applying a permutation to the coordinates of the points (X_1, X_2, X_3, X_4) of PG(3, 2) is an automorphism of PG(3, 2) stabilizing $Q^-(3,2)$. The stabilizer of $Q^-(3,2)$ inside $PGL(4,2)$ therefore contains a subgroup $S \cong S_4$. We will show that $B \in \mathcal{B}$. As $\mathcal{B}$ is stabilized by S, we are allowed to classify the sets B, up to isomorphisms in S.

The tangent plane π through the point $(1,1,1,1) \in Q^-(3,2)$ has equation $X_1 + X_2 + X_3 + X_4 = 0$. As B contains points outside π, i.e. points with weight 3, we may, without loss of generality, assume that $(1,1,1,0) \in B$. The external line $\{(1,1,0,0),(1,0,1,0),(0,1,1,0)\}$ must contain a point of B. Without loss of generality, we may assume that $(1,1,0,0) \in B$. We distinguish two cases.

Case 1: $(0,0,1,1) \notin B$. As each of the external lines $\{(0,0,1,1),(0,1,1,0),(0,1,0,1)\}$ and $\{(0,0,1,1),(1,0,1,0),(1,0,0,1)\}$ contains a point of B, at least one of $(0,1,1,0)$, $(0,1,0,1)$ belongs to B, as well as at least one of $(1,0,1,0)$, $(1,0,0,1)$. In the secant plane α with equation $X_4 = 0$, the points $(1,1,0,0)$ and $(1,1,1,0)$ belong to B, implying that at most one of the two remaining points $(0,1,1,0)$, $(1,0,1,0)$ in $\alpha \setminus Q^-(3,2)$ belongs to B. We will prove that precisely one of them belongs to B.

If this is not true, then none of $(0,1,1,0)$, $(1,0,1,0)$ belongs to B and we must have $(1,1,1,0),(1,1,0,0),(0,1,0,1),(1,0,0,1) \in B$. As the secant plane with equation $X_3 = 0$ contains at most three points of B (outside $Q^-(3,2)$), we have $(1,1,0,1) \notin B$. As each of the two external lines $\{(1,0,1,0),(0,1,1,1),(1,1,0,1)\}$ and $\{(0,1,1,0),(1,0,1,1),(1,1,0,1)\}$ contains a point of B, we then have that $(0,1,1,1)$ and $(1,0,1,1)$ belong to B. But that is impossible as it would imply that the tangent plane in the point $(0,0,1,0)$ with equation $X_1 + X_2 + X_4 = 0$ has all its points in B, with the exception of $(0,0,1,0) \in Q^-(3,2)$.

So, precisely one of the points $(0,1,1,0)$, $(1,0,1,0)$ belongs to B. Up to isomorphisms in S, we may assume that $(1,1,1,0),(1,1,0,0),(1,0,1,0)$ are in B and $(0,1,1,0) \notin B$. The external line $\{(0,1,1,0),(0,1,0,1),(0,0,1,1)\}$ meets B. As $(0,0,1,1)$ and $(0,1,1,0)$ are not in B, we have that $(0,1,0,1) \in B$. The four points outside $Q^-(3,2)$ in the secant plane $X_1 = X_3$ are $(1,1,1,0)$, $(1,0,1,0)$, $(0,1,0,1)$ and $(1,0,1,1)$. As not all these points can be contained in B, we have $(1,0,1,1) \notin B$. As the external line

$\{(0,1,1,0),(1,1,0,1),(1,0,1,1)\}$ contains points of B, we must have that $(1,1,0,1) \in B$. So, B_ω with $\omega = (\{1,2\},3,4)$ is contained in B. The minimality of B then implies that $B = B_\omega$.

Case 2: $(0,0,1,1) \in B$. So, $(1,1,1,0),(1,1,0,0)$ and $(0,0,1,1)$ are in B. The fourth point $(1,1,0,1)$ of $\mathrm{PG}(3,2)\setminus Q^-(3,2)$ in the secant plane through $(1,1,1,0)$, $(1,1,0,0)$, $(0,0,1,1)$ cannot belong to B. Next, considering the external lines $\{(1,1,0,1),(0,1,1,1),(1,0,1,0)\}$ and $\{(1,1,0,1),(1,0,1,1),(0,1,1,0)\}$ through $(1,1,0,1)$, we see that at least one of $(0,1,1,1)$, $(1,0,1,0)$ belongs to B, as well as at least one of $(1,0,1,1)$, $(0,1,1,0)$.

By considering the secant plane with equation $X_3 = X_4$ and taking into account that the points $(1,1,0,0)$, $(0,0,1,1)$ of B belong to that plane, we see that both $(0,1,1,1)$, $(1,0,1,1)$ cannot belong to B.

By considering the secant plane with equation $X_4 = 0$ and taking into account that the points $(1,1,1,0)$, $(1,1,0,0)$ of B belong to that plane, we see that both $(1,0,1,0)$, $(0,1,1,0)$ cannot belong to B.

So, we either have $(0,1,1,1),(0,1,1,0) \in B$ or $(1,0,1,1),(1,0,1,0) \in B$. So, B_ω with ω equal to either $(\{2,3\},1,4)$ or $(\{1,3\},2,4)$ is contained in B. The minimality of B then implies that $B = B_\omega$.

3. Proofs of Theorems 1.7 and 1.8

3.1. **Preliminaries.** Let B be an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10. If B is not minimal, then $B = B' \cup \{x\}$, where B' is an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 9 and x is a point of $\mathrm{PG}(3,3)$ not belonging to B'. By Proposition 1.2, $B' = \pi \setminus Q^-(3,3)$ for some secant plane π of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$. We then have case (1) of Theorem 1.8. Therefore, we may assume that B is minimal. We divide the treatment into two cases:

- **Case I**: There exists a secant plane π of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ such that all the points of π exterior to the conic $\pi \cap Q^-(3,3)$ are contained in B.
- **Case II**: There does not exist any secant plane π of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ such that all the points of π exterior to $\pi \cap Q^-(3,3)$ are contained in B.

We will prove in Section 3.2 that the set Y defined in Theorem 1.7 is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10 (Proposition 3.5) and that if Case I occurs, then B is isomorphic to Y (Proposition 3.6). We shall then prove in Section 3.3 that there are no examples of blocking sets corresponding to Case II. We repeatedly use the following lemma, mostly without mention.

Lemma 3.1. *We have $B \cap Q^-(3,3) = \emptyset$.*

Proof. This follows from the minimality of B and the fact that $B \setminus Q^-(3,3)$ is also an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$. □

3.2. **Treatment of Case I.** In this subsection, we suppose that B is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10 containing the set E_π of all points of a secant plane π that are exterior with respect to the conic $\mathcal{C}_\pi := \pi \cap Q^-(3,3)$ in π. Let I_π denote the set of three points of π that are interior with respect to $\mathcal{C}_\pi$. Through each point of I_π, there are four lines belonging to $\mathcal{E}^-$ that are not lines of π (see Section 1.2).

Lemma 3.2. *The following hold:*

(1) $B \cap \pi = E_\pi$ *and* $|B \setminus \pi| = 4$.
(2) *If* $x \in I_\pi$*, then each of the four external lines through* x *not contained in* π *meets* B *in precisely one point.*

Proof. Since $\pi \setminus \mathcal{C}_\pi$ is an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 9 containing E_π, the minimality of B with $|B| = 10$ implies that there must exist a point in I_π not belonging to B. Now, let x be an arbitrary point of $I_\pi \setminus B$. Since each of the four external lines through x not contained in π meets B, we must have $|B \setminus \pi| \geq 4$. As $|B| = 10$ and E_π is contained in B with $|E_\pi| = 6$, we thus see that the conclusions of the lemma must hold. □

Let Q denote the quadratic form defining the quadric $Q^-(3,3)$. We can choose a reference system in $\mathrm{PG}(3,3)$ such that $Q^-(3,3)$ has equation

$$Q(X_1, X_2, X_3, X_4) = X_1X_2 + X_3^2 + X_4^2 = 0$$

and π has equation $X_4 = 0$. We then have:

$$\begin{aligned}
\mathcal{C}_\pi &= \{\langle(1,0,0,0)\rangle, \langle(0,1,0,0)\rangle, \langle(1,-1,1,0)\rangle, \langle(1,-1,-1,0)\rangle\},\\
E_\pi &= \{\langle(0,0,1,0)\rangle, \langle(1,1,0,0)\rangle, \langle(1,0,1,0)\rangle, \langle(1,0,-1,0)\rangle,\\
&\qquad\qquad\qquad\qquad \langle(0,1,1,0)\rangle, \langle(0,1,-1,0)\rangle\},\\
I_\pi &= \{\langle(1,-1,0,0)\rangle, \langle(1,1,1,0)\rangle, \langle(1,1,-1,0)\rangle\}.
\end{aligned}$$

In fact, $\mathcal{C}_\pi$ (respectively, E_π, I_π) consists of all points $\langle(X_1, X_2, X_3, 0)\rangle$ of π for which $Q(X_1, X_2, X_3, 0)$ is equal to 0 (respectively, 1, -1). If $f : \mathbb{F}_3^4 \times \mathbb{F}_3^4 \to \mathbb{F}_3$ is the symmetric bilinear form associated with Q, then

$$f((X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4)) = X_1Y_2 + X_2Y_1 - X_3Y_3 - X_4Y_4$$

for all $(X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4) \in \mathbb{F}_3^4$.

Lemma 3.3. *If* $\langle(a,b,c,1)\rangle \in B \setminus \pi$*, then* $(a,b,c) \neq (0,0,0)$ *and* $\langle(a,b,c,0)\rangle \in \mathcal{C}_\pi$.

Proof. Suppose that $(a,b,c) = (0,0,0)$ and let $\langle(u,v,w,0)\rangle$ be an arbitrary point of I_π. As $Q(u,v,w,1) = Q(u,v,w,0) + Q(0,0,0,1) = -1 + 1 = 0$, the line through $\langle(a,b,c,1)\rangle = \langle(0,0,0,1)\rangle$ and $\langle(u,v,w,0)\rangle$ would not be external as required by Lemma 3.2. Hence, $(a,b,c) \neq (0,0,0)$.

We also have $\langle(a,b,c,0)\rangle \notin I_\pi$ as otherwise $Q(a,b,c,1) = Q(a,b,c,0) + Q(0,0,0,1) = -1 + 1 = 0$, in contradiction with $B \cap Q^-(3,3) = \emptyset$ (Lemma 3.1).

Suppose that $\langle(a,b,c,0)\rangle \in E_\pi$. Consider then the unique external line through $\langle(a,b,c,0)\rangle$ contained in π. This external line contains precisely two points of I_π, say $\langle(u_1,v_1,w_1,0)\rangle$ and $\langle(u_2,v_2,w_2,0)\rangle$. Without loss of generality, we may suppose that we have chosen (u_1,v_1,w_1) and (u_2,v_2,w_2) in such a way that $(a,b,c)=(u_2,v_2,w_2)+\lambda(u_1,v_1,w_1)$ for some $\lambda\in\mathbb{F}_3$. The line through the points $\langle(a,b,c,1)\rangle \in B\setminus\pi$ and $\langle(u_1,v_1,w_1,0)\rangle\in I_\pi$ then contains the point $\langle(u_2,v_2,w_2,1)\rangle$ belonging to $Q^-(3,3)$ as $Q(u_2,v_2,w_2,1)=Q(u_2,v_2,w_2,0)+Q(0,0,0,1)=-1+1=0$. But that is impossible as such a line must belong to $\mathcal{E}^-$ by Lemma 3.2. $\square$

Lemma 3.4. *Let $(a_1,b_1,c_1,0)$, $(a_2,b_2,c_2,0)$, $(a_3,b_3,c_3,0)$ and $(a_4,b_4,c_4,0)$ be mutually distinct elements of $\mathbb{F}_3^4$ such that $\langle(a_i,b_i,c_i,0)\rangle$ is a point of $\mathcal{C}_\pi$ for each $i\in\{1,2,3,4\}$. Then*

$$X:=\{\langle(a_i,b_i,c_i,1)\rangle \mid i\in\{1,2,3,4\}\}\cup E_\pi$$

is an $\mathcal{E}^-$-blocking set in PG$(3,3)$ *if and only if $\langle(a_i-a_j,b_i-b_j,c_i-c_j,0)\rangle\notin I_\pi$ for all $i,j\in\{1,2,3,4\}$ with $i\neq j$.*

Proof. We first prove that if $\langle(a,b,c,0)\rangle\in\mathcal{C}_\pi$, then $\langle(a,b,c,1),(a',b',c',0)\rangle$ is an external line for every $\langle(a',b',c',0)\rangle\in I_\pi$. Indeed, we have $Q(a,b,c,1)=Q(a,b,c,0)+Q(0,0,0,1)=0+1\neq 0$ and $Q(a',b',c',0)=-1$. The secant line $\langle(a,b,c,0),(a',b',c',0)\rangle$ in π contains only one point of I_π, namely $\langle(a',b',c',0)\rangle$. If $(x,y,z)\in\{(a+a',b+b',c+c'),(a-a',b-b',c-c')\}$, then $\langle(x,y,z,0)\rangle$ is a point of $\langle(a,b,c,0),(a',b',c',0)\rangle$ not belonging to I_π and hence the point $\langle(x,y,z,1)\rangle\in\langle(a,b,c,1),(a',b',c',0)\rangle$ does not belong to $Q^-(3,3)$ as $Q(x,y,z,1)=Q(x,y,z,0)+Q(0,0,0,1)=Q(x,y,z,0)+1\neq 0$.

Now, X is an $\mathcal{E}^-$-blocking set in PG$(3,3)$ if and only if for every point $\langle(a',b',c',0)\rangle\in I_\pi$, the four lines $\langle(a',b',c',0),(a_i,b_i,c_i,1)\rangle$, $i\in\{1,2,3,4\}$, are distinct (external) lines. The latter statement precisely holds when the condition in the lemma is satisfied. $\square$

As an application of Lemma 3.4, we have the following proposition that also proves Theorem 1.7.

Proposition 3.5. *The sets*

$$Y:=\{\langle(1,0,0,1)\rangle,\langle(0,-1,0,1)\rangle,\langle(1,-1,1,1)\rangle,\langle(1,-1,-1,1)\rangle\}\cup E_\pi$$

and

$$Z:=\{\langle(-1,0,0,1)\rangle,\langle(0,1,0,1)\rangle,\langle(-1,1,-1,1)\rangle,\langle(-1,1,1,1)\rangle\}\cup E_\pi$$

are isomorphic minimal $\mathcal{E}^-$-blocking sets in PG$(3,3)$ *of size* 10.

Proof. Clearly, $|Y|=10=|Z|$. The fact that Y and Z are $\mathcal{E}^-$-blocking sets in PG$(3,3)$ follows from Lemma 3.4. Since the map

$$(X_1,X_2,X_3,X_4)\mapsto(X_1,X_2,X_3,-X_4)$$

defines an automorphism of PG$(3,3)$ stabilizing $Q^-(3,3)$ and fixing π pointwise, it follows that Y and Z are isomorphic.

If Y and Z were not minimal as $\mathcal{E}^-$-blocking sets, then they would be of the form $(\pi' \setminus Q^-(3,3)) \cup \{x\}$, where π' is a secant plane of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ and x is a point not belonging to $\pi' \setminus Q^-(3,3)$. This is obviously not the case here. □

The following proposition proves Theorem 1.8 if Case I occurs.

Proposition 3.6. *B is equal to either Y or Z.*

Proof. Let $\mathcal{G}$ be the graph with vertex set

$$V(\mathcal{G}) = \{(1,0,0,1),(0,-1,0,1),(1,-1,1,1),(1,-1,-1,1),\\ (-1,0,0,1),(0,1,0,1),(-1,1,-1,1),(-1,1,1,1)\} \subseteq \mathbb{F}_3^4,$$

where two distinct vertices $(a,b,c,1)$ and $(a',b',c',1)$ are adjacent whenever $\langle(a-a',b-b',c-c',0)\rangle \in I_\pi \cup \mathcal{C}_\pi$. Then $\mathcal{G}$ is isomorphic to the complete bipartite graph $K_{4,4}$ with the two parts

$$\{(1,0,0,1),(0,-1,0,1),(1,-1,1,1),(1,-1,-1,1)\}$$

and

$$\{(-1,0,0,1),(0,1,0,1),(-1,1,-1,1),(-1,1,1,1)\}$$

for which the set of all edges $\{(a,b,c,1),(a',b',c',1)\}$ satisfying $\langle(a-a',b-b',c-c',0)\rangle \in \mathcal{C}_\pi$ is a complete (that is, perfect) matching.

We know that $B \setminus E_\pi$ is a subset of $\{\langle(a,b,c,1)\rangle : (a,b,c,1) \in V(\mathcal{G})\}$ by Lemma 3.3 and that B must be obtained as in Lemma 3.4. The first paragraph of this proof in combination with Lemma 3.4 then immediately implies that B is either Y or Z. □

3.3. **Treatment of Case II.** In this subsection, we suppose that B is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10 satisfying the following:

(∗) There is no secant plane π of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ such that all the points of π exterior to the conic $\pi \cap Q^-(3,3)$ are contained in B.

We shall derive a contradiction at the end of this section.

Lemma 3.7. *Let π be a tangent plane of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$. Then $|\pi \cap B| \geq 3$, with equality if and only if $\pi \cap B = L \setminus Q^-(3,3)$ for some tangent line L in π.*

Proof. This is a special case of a more general result proved in [3, Proposition 3.1], also see [13, Lemma 2.4]. □

Lemma 3.8. *A secant plane of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ cannot contain more than five points of B.*

Proof. Suppose that there is a secant plane π containing at least six points of B. By our assumption (∗), there exists a point x in $\pi \setminus B$ exterior with respect to the conic $\pi \cap Q^-(3,3)$. Through x, there are five external lines not contained in π. Each of these five external lines contains an extra point of B, implying that $|B| \geq |\pi \cap B| + 5 \geq 6 + 5 = 11$, a contradiction. □

Lemma 3.9. *There does not exist any tangent plane of* PG$(3,3)$ *with respect to* $Q^-(3,3)$ *meeting* B *in an external line.*

Proof. Suppose π_1 is a tangent plane meeting B in an external line L. Let π_2 be the other tangent plane through L, and π_3, π_4 be the two secant planes through L. Each of $\pi_3 \setminus L$ and $\pi_4 \setminus L$ contains at most one point of B by Lemma 3.8, implying that $\pi_2 \setminus L$ contains at least four points of B. Thus, the tangent plane π_2 contains at least eight points of B.

Now, let x be a point of $\pi_2 \setminus Q^-(3,3)$ not contained in B. There are three external lines through x not contained in π. Each of these three external lines contains a point of B, implying that $|B| \geq |\pi_2 \cap B| + 3 \geq 8 + 3 = 11$, a contradiction. □

Lemma 3.10. *There does not exist any tangent plane of* PG$(3,3)$ *with respect to* $Q^-(3,3)$ *meeting* B *in three points.*

Proof. Suppose π is a tangent plane meeting B in three points. Then there exists a tangent line L in π such that $\pi \cap B = L \setminus Q^-(3,3)$ by Lemma 3.7.

By Lemma 3.8, each of the three secant planes through L contains besides the points of $L \setminus Q^-(3,3)$ at most two other points of B, giving that $|B| \leq |L \setminus Q^-(3,3)| + 3 \cdot 2 = 9$, a contradiction. □

Lemma 3.11. *Each tangent plane of* PG$(3,3)$ *with respect to* $Q^-(3,3)$ *intersects* B *in precisely four points.*

Proof. We count in two ways the number N of pairs (x, π), where $x \in B$ and π is a tangent plane through x. As there are 10 tangent planes, we know by Lemmas 3.7 and 3.10 that $N \geq 40$, with equality if and only if every tangent plane contains precisely four points. On the other hand, we have $|B| = 10$ possibilities for x and we know by Lemma 3.1 that there are four possibilities for π for a given x. So, $N = 40$ and every tangent plane contains precisely four points of π. □

Lemma 3.12. *Let* x *be a point of* $Q^-(3,3)$ *and* π *be the tangent plane of* PG$(3,3)$ *with respect to* $Q^-(3,3)$ *through* x*. Then*

$$\pi \cap B = (L_x \setminus Q^-(3,3)) \cup \{z_x\}$$

for some (tangent) line L_x *of* π *through* x *and some point* $z_x \in \pi \setminus L_x$*.*

Proof. By Lemma 3.11, we have $|\pi \cap B| = 4$. First suppose that every tangent line contained in π meets B. Then $\pi \cap B$ is a blocking set of size 4 with respect to all lines of π and so $\pi \cap B$ is a line of π by the classical result of Bose and Burton [4, Theorem 1]. As $B \cap Q^-(3,3) = \emptyset$, $\pi \cap B$ must be an external line, in contradiction with Lemma 3.9.

Therefore, there is some tangent line in π disjoint from B. As $|B \cap \pi| = 4$, this implies that there is some tangent line L_x in π meeting B in at least two points, say x_1 and x_2. Let x_3 denote the unique point in $L_x \setminus Q^-(3,3)$ distinct from x_1 and x_2. If $x_3 \notin B$, then each of the three external lines through x_3 contained in π would contain another point of B, implying $|\pi \cap B| \geq 5$

which is not possible. So, $x_3 \in B$ and $L_x \setminus Q^-(3,3)$ is contained in B. Since $|\pi \cap B| = 4$ and $B \cap Q^-(3,3) = \emptyset$, there exists a unique point $z_x \in \pi \cap B$ not belonging to L_x. Thus, $\pi \cap B = (L_x \setminus Q^-(3,3)) \cup \{z_x\}$. □

Lemma 3.13. *There exist no four mutually distinct points x_1, x_2, x_3, x_4 in $Q^-(3,3)$ such that the tangent lines L_{x_1}, L_{x_2}, L_{x_3}, L_{x_4} obtained as in Lemma 3.12 meet in a common point.*

Proof. Suppose L_{x_1}, L_{x_2}, L_{x_3}, L_{x_4} meet in a common point y. Then the secant plane y^τ contains the points x_1, x_2, x_3 and x_4. The number of points of B contained in $L_{x_1} \cup L_{x_2} \cup L_{x_3} \cup L_{x_4}$ equals 9. There are also at least two points of B in y^τ which follows from Proposition 1.1. This would imply that $|B| \geq 11$, a contradiction. □

Lemma 3.14. *Each point of B is contained in precisely three tangent lines L_x, $x \in Q^-(3,3)$.*

Proof. We count in two ways the number N of pairs (x, y) with $x \in Q^-(3,3)$ and $y \in L_x \setminus Q^-(3,3) \subseteq B$. As $|Q^-(3,3)| = 10$, there are $N = 30$ such pairs. On the other hand, there are at most $|B| = 10$ possibilities for y. For a given y, there are at most three possibilities for x with $y \in L_x$ by Lemma 3.13. As $N = 30$, we thus see that every point of B is contained in precisely three lines L_x, $x \in Q^-(3,3)$. □

Derivation of a contradiction:

Fix a point $x \in B$ and let $L_{u_1}, L_{u_2}, L_{u_3}$ be the three mutually distinct tangent lines through x obtained in Lemma 3.14, where $u_1, u_2, u_3 \in Q^-(3,3)$. Put

$$L_{u_1} := \{x, y_1, z_1, u_1\},\ L_{u_2} := \{x, y_2, z_2, u_2\} \text{ and } L_{u_3} := \{x, y_3, z_3, u_3\}.$$

Then $\{x, y_1, z_1, y_2, z_2, y_3, z_3\} \subseteq B$. We thus have already found seven points of B. Each of the planes $\langle L_{u_1}, L_{u_2}\rangle$, $\langle L_{u_1}, L_{u_3}\rangle$ and $\langle L_{u_2}, L_{u_3}\rangle$ is a secant plane containing, by Lemma 3.8, no further points of B than those in $\{x, y_1, z_1, y_2, z_2, y_3, z_3\}$. So, for the point $y_1 \in B$, if

$$L_{u_1} = \{y_1, x, z_1, u_1\},\ L_{u_2'} := \{y_1, y_2', z_2', u_2'\} \text{ and } L_{u_3'} := \{y_1, y_3', z_3', u_3'\}$$

are the three mutually distinct tangent lines through y_1 obtained in Lemma 3.14 with $u_2', u_3' \in Q^-(3,3)$ and $y_2', z_2', y_3', z_3' \in B$, then none of the points y_2', z_2', y_3', z_3' can be contained in $\{x, y_1, z_1, y_2, z_2, y_3, z_3\}$. This would imply that $\{x, y_1, z_1, y_2, z_2, y_3, z_3, y_2', z_2', y_3', z_3'\}$ is a set of 11 points of B, a contradiction to the fact that B contains exactly 10 points. This proves that Case II does not occur.

4. Proof of Theorem 1.9

In this section, w is a point of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$, π is the secant plane w^τ of $\mathrm{PG}(3,3)$, where τ is the orthogonal polarity associated with $Q^+(3,3)$, and $B = E_\pi \cup \{w\}$.

Proof of Theorem 1.9. Clearly, we have $|B| = 7$. We need to show that B is a minimal $\mathcal{E}^+$-blocking set in PG$(3,3)$. Through w, there are:

- four outer tangents, namely the lines wx with x a point in the conic $\pi \cap Q^+(3,3)$;
- three external lines with respect to $Q^+(3,3)$, namely the lines $wz \in \mathcal{E}^+$ with $z \in I_\pi$ (see [8, Corollary 2.4]), where I_π is the set of three points of π that are interior with respect to $\pi \cap Q^+(3,3)$;
- six secant lines with respect to $Q^+(3,3)$, namely the lines wy with $y \in E_\pi$.

Through a point $u \in I_\pi$, there are three lines belonging to $\mathcal{E}^+$. Two of them are contained in π and so each of them meets E_π. The remaining one external line necessarily coincides with uw.

Now, any external line $L \in \mathcal{E}^+$ is either contained in π, meets π in a point of E_π or meets π in a point of I_π. In the first case, L contains two points of E_π. In the second case, L contains one point of E_π but not the point w. In the third case, L contains the point w but not any of the points of E_π.

It follows from the above that B is an $\mathcal{E}^+$-blocking set in PG$(3,3)$, but also that it is minimal as removing one point x of B would imply the existence of lines belonging to $\mathcal{E}^+$ that are disjoint from $B \setminus \{x\}$. Indeed, if $x = w$, then this would be the case for the external lines $wz \in \mathcal{E}^+$ with $z \in I_\pi$. If $x \in E_\pi$, then this would be the case for the two external lines through x not contained in π. This completes the proof. □

Now, let L^* be an outer tangent contained in π with tangency point x^* in $Q^+(3,3)$. As wx^* is also an outer tangent, the plane $\pi^* := \langle w, L^* \rangle$ is a tangent plane with tangency point x^*. Further, π^* meets B in four points among which three are on the same outer tangent L^* and the remaining one point, namely w is on the other outer tangent through x^*, that is, $\pi^* \cap B = (L^* \setminus \{x^*\}) \cup \{w\}$.

Lemma 4.1. *Any $\mathcal{E}^+$-blocking set in* PG$(3,3)$ *of size* 7 *intersecting π^* in $B \cap \pi^*$ coincides with B.*

Proof. Let X be an $\mathcal{E}^+$-blocking set in PG$(3,3)$ of size 7 such that $X \cap \pi^* = B \cap \pi^*$. We claim that $X = B$. It suffices to prove that there is at most one such $\mathcal{E}^+$-blocking set X.

Put $wx^* := \{w, x^*, u_1, u_2\}$. Then $w \in B$ and x^*, u_1, u_2 are not contained in B. Through the outer tangent wx^*, there are three secant planes, say π_1, π_2, π_3, of PG$(3,3)$ with respect to $Q^+(3,3)$. For $i \in \{1,2,3\}$, the set $X \cap \pi_i$ is a blocking set in π_i of the external lines with respect to the conic $\mathcal{C}_i = \pi_i \cap Q^+(3,3)$. Then $|X \cap \pi_i| \geq 2$ by Proposition 1.1, and so each π_i contains at least one extra point of X besides w. But as $|X| = 7$ and $|X \cap \pi^*| = |B \cap \pi^*| = 4$, there are precisely three points in $X \setminus \pi^*$, showing that each π_i contains precisely one extra point of X different from w. It suffices to show that this extra point in each π_i is uniquely determined.

As wx^* is tangent to the conic $\mathcal{C}_i$, the points u_1 and u_2 of π_i are exterior with respect to $\mathcal{C}_i$. So, each u_j with $j \in \{1,2\}$ is contained in a unique line M_{ij} of π_i external to $\mathcal{C}_i$. Each of the lines M_{i1} and M_{i2} contains a point of X. As π_i contains only one extra point of X besides w, this extra point of X in π_i must coincide with $M_{i1} \cap M_{i2}$. □

We thus have also shown the following.

Corollary 4.2. *The $\mathcal{E}^+$-blocking sets in* PG(3, 3) *of size* 7 *disjoint from* $Q^+(3,3)$ *and intersecting a tangent plane in a set of four points three of which are on the same outer tangent and the remaining one is on the other outer tangent are precisely the $\mathcal{E}^+$-blocking sets described in Theorem 1.9.*

5. Proof of Theorem 1.10

Let B be an $\mathcal{E}^+$-blocking set in PG(3, 3) of size 7. If B is not minimal, then $B = B' \cup \{x\}$, where B' is an $\mathcal{E}^+$-blocking set in PG(3, 3) of size 6 and x is a point not belonging to B'. By Proposition 1.3, $B' = \pi \setminus Q^+(3,3)$ for some tangent plane π of PG(3, 3) with respect to $Q^+(3,3)$. We then have case (1) of Theorem 1.10. So, we may assume the following:

Assumption 1. B is minimal.

As $B \setminus Q^+(3,3)$ is also an $\mathcal{E}^+$-blocking set in PG(3, 3), we have the following by the minimality of B.

Lemma 5.1. $B \cap Q^+(3,3) = \emptyset$.

Each tangent plane of PG(3, 3) with respect to $Q^+(3,3)$ meets B thus in at most six points.

Lemma 5.2. *There is no tangent plane of* PG(3, 3) *with respect to* $Q^+(3,3)$ *meeting B in precisely six points.*

Proof. If this were the case, then B would be as in (1) of Theorem 1.10, contrary to the assumption that B is minimal. □

Lemma 5.3. *There exists no tangent plane of* PG(3, 3) *with respect to* $Q^+(3,3)$ *intersecting B in precisely five points.*

Proof. Suppose π is a tangent plane intersecting B in precisely five points and let x be the unique point in $\pi \setminus (B \cup Q^+(3,3))$. Through x, there are three lines belonging to $\mathcal{E}^+$ not contained in π. Each of these lines contains at least one point of B, showing that $|B| = |B \cap \pi| + |B \setminus \pi| \geq 5 + 3 = 8$, a contradiction. □

We may also make the following assumption.

Assumption 2. There is no tangent plane of PG(3, 3) with respect to $Q^+(3,3)$ intersecting B in a set of four points, three of which are on the same outer tangent and the remaining one is on the other outer tangent.

Indeed, if this were not the case, then Corollary 4.2 implies that we would have case (2) of Theorem 1.10. We shall derive a contradiction at the end of this section under Assumptions 1 and 2.

Lemma 5.4. *There exists no tangent plane of* PG$(3,3)$ *with respect to* $Q^+(3,3)$ *intersecting* B *in precisely four points.*

Proof. Suppose π is a tangent plane intersecting B in precisely four points. By Lemma 5.1 and Assumption 2, we then know that each of the two outer tangents in π contains precisely two points of B. So, there exists a unique line L in π that is secant to $Q^+(3,3)$ and disjoint from B. There are two secant planes, say π_1, π_2, of PG$(3,3)$ with respect to $Q^+(3,3)$ through L. Each of them contains at least two points of B by Proposition 1.1, showing that $|B| \geq |B \cap \pi| + |B \cap \pi_1| + |B \cap \pi_2| \geq 4+2+2 = 8$, a contradiction. □

Lemma 5.5. *There exists no outer tangent intersecting* B *in precisely three points.*

Proof. Suppose L is an outer tangent meeting B in precisely three points. There is a unique tangent plane π through L and π contains besides L one other outer tangent L'. By Lemmas 5.1, 5.2, 5.3 and 5.4, we know that $L' \cap B = \emptyset$. There are three secant planes, say π_1, π_2, π_3, of PG$(3,3)$ with respect to $Q^+(3,3)$ through L'. Each of them contains at least two points of B by Proposition 1.1, giving that $|B| \geq |B \cap \pi| + |B \cap \pi_1| + |B \cap \pi_2| + |B \cap \pi_3| \geq 3+2+2+2 = 9$, a contradiction. □

Lemma 5.6. *There exists no tangent plane of* PG$(3,3)$ *with respect to* $Q^+(3,3)$ *meeting* B *in precisely three points.*

Proof. Suppose π is a tangent plane with tangency point x meeting B in precisely three points. By Lemma 5.5, there exists a unique outer tangent L_1 in π containing precisely two points of B and the other outer tangent L_2 in π contains a unique point of B. By Lemma 4.1, we know that there exists a unique $\mathcal{E}^+$-blocking set B^* in PG$(3,3)$ of size 7 that meets π in $(L_1 \setminus \{x\}) \cup (L_2 \cap B)$.

Put $L_2 \cap B = \{y\}$ and $L_2 = \{x, u_1, u_2, y\}$. There are three secant planes, say π_1, π_2, π_3, of PG$(3,3)$ with respect to $Q^+(3,3)$ through L_2. Each of them contains, besides the point y, at least one extra point of B by Proposition 1.1. As $((\pi_1 \cup \pi_2 \cup \pi_3) \setminus L_2) \cap B$ has size $|B| - |B \cap \pi| = 4$, two of these planes, say π_1 and π_2, contain precisely one extra point of B, while the remaining plane π_3 contains exactly two extra points of B (besides y). By the proof of Lemma 4.1, we know that each plane π_i, $i \in \{1,2,3\}$, contains a unique point $z_i \in B^*$ not belonging to L_2 (that is, $z_i \neq y$). If M_{ij} denotes the unique external line in π_i through the point u_j with $j \in \{1,2\}$, then z_i is obtained as the intersection point of M_{i1} and M_{i2}. Each of the external lines M_{i1} and M_{i2} must also contain a point of B, showing that z_1 and z_2 are the unique points of B in respectively $\pi_1 \setminus L_2$ and $\pi_2 \setminus L_2$. We claim that z_3 is not a point of B.

By the original construction of the $\mathcal{E}^+$-blocking set B^* (see Theorem 1.9, Lemma 4.1 and its proof), we know that $B^* = \{y\} \cup (L_1 \setminus \{x\}) \cup \{z_1, z_2, z_3\}$, where $(L_1 \setminus \{x\}) \cup \{z_1, z_2, z_3\}$ consists of the six points of $\pi^* := y^\tau$ that are exterior with respect to the conic $\pi^* \cap Q^+(3,3)$.

Let z' denote the unique point of $L_1 \setminus (\{x\} \cup B)$. We know that there are at least four points of B in π^*, namely z_1, z_2 and the two points in $L_1 \setminus \{x, z'\}$. We also know that there are at least three points of B outside π^*, namely y and two points on the two external lines through z' not contained in π^* (note that yz' is a secant line). These must constitute all the seven points of B and it follows that z_3 is not a point of B.

Thus, z' and z_3 are the only points of B^* not contained in B. Consider the line $z'z_3$ in π^*. It cannot be a secant line as both z' and z_3 are exterior points with respect to $\pi^* \cap Q^+(3,3)$. If $z'z_3$ is an external line containing besides z' and z_3 two interior points with respect to $\pi^* \cap Q^+(3,3)$, then this external line would be disjoint from B, which is impossible. So, $z'z_3$ is an outer tangent in π^* and it must contain one of z_1 and z_2, say z_1, as the third exterior point with respect to $\pi^* \cap Q^+(3,3)$. The exterior point z_1 is contained in two outer tangents of π^*. One of them is $z'z_3$. The other outer tangent through z_1 contains precisely three points of B, namely z_1, z_2 and one point of $L_1 \cap B$. But that is in contradiction with Lemma 5.5. This completes the proof. □

Lemma 5.7. *Every outer tangent intersects B in either 0 or 1 point.*

Proof. Suppose that this is not the case. Then by Lemmas 5.1 and 5.5, there exists an outer tangent L meeting B in exactly two points. Let π be the unique tangent plane containing L. By Lemmas 5.1, 5.2, 5.3, 5.4 and 5.6, the other outer tangent L' in π is disjoint from B. Then each of the three secant planes, say π_1, π_2, π_3, through L' contains at least two points of B by Proposition 1.1, showing that $|B| \geq |B \cap \pi| + |B \cap \pi_1| + |B \cap \pi_2| + |B \cap \pi_3| \geq 2+2+2+2 = 8$, a contradiction. □

Lemma 5.8. *The number of outer tangents disjoint from B is equal to 4. The number of outer tangents intersecting B in a singleton is equal to 28.*

Proof. Let λ_i with $i \in \{0, 1\}$ denote the number of outer tangents meeting B in exactly i points. Each point of $Q^+(3,3)$ is contained in precisely two outer tangents and so the total number of outer tangents is equal to $|Q^+(3,3)| \cdot 2 = 16 \cdot 2 = 32$. By Lemma 5.7, we thus have $\lambda_0 + \lambda_1 = 32$. As each point of B is outside $Q^+(3,3)$ by Lemma 5.1 and is contained in four outer tangents, counting the pairs (x, L) with L an outer tangent and $x \in L \cap B$ yields $\lambda_1 = 7 \cdot 4 = 28$. We then find $\lambda_0 = 32 - \lambda_1 = 4$. □

Lemma 5.9. *If π is a secant plane of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$, then among the six points of π that are exterior with respect to the conic $\pi \cap Q^+(3,3)$ at most two can belong to B.*

Proof. If B contains at least three of the six exterior points of π, then there would exist an outer tangent in π containing at least two of these points of B, contradicting Lemma 5.7. □

The following is a consequence of Lemmas 5.1 and 5.9.

Corollary 5.10. *If π is a secant plane of* $\mathrm{PG}(3,3)$ *with respect to $Q^+(3,3)$, then $|\pi \cap B| \leq 5$. If $|\pi \cap B| = 5$, then all three points of π interior to the conic $\pi \cap Q^+(3,3)$ are contained in B and precisely two of the six points of π exterior to $\pi \cap Q^+(3,3)$ are contained in B.*

Lemma 5.11. *There cannot exist secant planes of* $\mathrm{PG}(3,3)$ *with respect to $Q^+(3,3)$ meeting B in precisely five points.*

Proof. Suppose π is a secant plane meeting B in precisely five points. Let y_1 and y_2 denote the two points of B not contained in π. By Corollary 5.10, there exist four points x_1, x_2, x_3, x_4 in $\pi \setminus B$ that are exterior with respect to the conic $\pi \cap Q^+(3,3)$. Through each x_i, $i \in \{1,2,3,4\}$, there are two lines belonging to $\mathcal{E}^+$ not contained in π. As each of these lines contains a point of B, they have to coincide with x_iy_1 and x_iy_2. It follows that y_1x_1, y_1x_2, y_1x_3 and y_1x_4 are four external lines through y_1, in contradiction with the fact that there are only three external lines through y_1. □

Lemma 5.12. *There exists no tangent plane of* $\mathrm{PG}(3,3)$ *with respect to $Q^+(3,3)$ meeting B in precisely two points.*

Proof. Suppose π is a tangent plane intersecting B in precisely two points. By Lemma 5.7, each of the two outer tangents in π contains one point of B. Let L denote the secant line with respect to $Q^+(3,3)$ through the two points of $\pi \cap B$. We count the points of B contained in the four planes through L. By Lemmas 5.1, 5.2, 5.3, 5.4 and 5.6, each tangent plane contains at most two points of B. This implies that each of the two tangent planes through L contains precisely two points of B, namely the two points of $L \cap B = \pi \cap B$. By Corollary 5.10 and Lemma 5.11, each of the two secant planes through L contains at most four points of B (including the two points of $L \cap B$). It follows that $|B| \leq 6$, a contradiction. □

By Lemmas 5.1, 5.2, 5.3, 5.4, 5.6 and 5.12, we thus have the following:

Corollary 5.13. *Every tangent plane of* $\mathrm{PG}(3,3)$ *with respect to $Q^+(3,3)$ contains either* 0 *or* 1 *point of B.*

Derivation of a contradiction: We are now ready to derive a contradiction (under Assumptions 1 and 2). Let n_i with $i \in \{0,1\}$ denote the number of tangent planes meeting B in precisely i points. By Corollary 5.13, we have $n_0 + n_1 = 16$ and so $n_1 \leq 16$. The total number of outer tangents meeting B in a singleton is equal to n_1. Then $n_1 \leq 16 < 28$ contradicts Lemma 5.8.

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