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ON A CONSTRUCTION OF SELF-ORTHOGONAL CODES FROM ORBIT MATRICES OF 2-DESIGNS

MARIJA MAKSIMOVIĆ AND SANJA RUKAVINA

Abstract. In 2003, Harada and Tonchev presented a construction of self-orthogonal codes from orbit matrices of symmetric 2-designs with fixed point free automorphisms. Since then, the constructions of self-orthogonal codes from orbit matrices of 2-designs has been extensively studied. In this paper, we present new constructions of self-orthogonal codes from orbit matrices of 2-designs for the cases not covered by the previously described methods. We construct self-orthogonal codes from orbit matrices of 2-$(1024, 496, 249)$ and 2-$(45, 5, 1)$ designs. Some of the constructed codes are optimal.

1. Introduction and preliminaries

In [12], Harada and Tonchev presented a construction of self-orthogonal codes from orbit matrices of symmetric 2-designs with fixed point free automorphisms. Since then, the constructions of self-orthogonal codes from orbit matrices of incidence structures, in particular designs, has been extensively studied ([5], [7], [8], [9]). In this paper, we present new methods for a construction of self-orthogonal codes from orbit matrices of 2-designs for the cases not covered by the previously described methods. As a demonstration of the constructions presented, we construct self-orthogonal codes from 2-$(45, 5, 1)$ and 2-$(1024, 496, 249)$ designs.

We assume that the reader is familiar with the basic terminology from design theory and coding theory. For background information, we refer the reader to [1] and [16].

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$, where $|\mathcal{P}| = v$, $|\mathcal{B}| = b$, each block $B \in \mathcal{B}$ is incident with exactly k points, every pair of distinct points from $\mathcal{P}$ is incident with exactly λ blocks and each point is incident with exactly r blocks is a

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2-(v, b, r, k, λ) *design* or a 2-(v, k, λ) *design*. An *automorphism group of the design* $\mathcal{D}$, in the notation $Aut(D)$, is the group $G = \{g \in Sym(\mathcal{P}) : \mathcal{B}^g = \mathcal{B}\}$.

A *linear q-ary $[n, k]$ code* C over the finite field $\mathbb{F}_q$ of prime-power order q is a k-dimensional subspace of the n-dimensional vector space over $\mathbb{F}_q$. The *weight* of a codeword is the number of its elements that are nonzero and the distance between two codewords is the *Hamming distance* between them, i.e., the number of elements in which they differ. The *minimum distance* between different codewords is denoted by d. The minimum distance of a linear code is the minimum weight of its nonzero codewords. If a linear code C over a field of order q has length n, dimension k, and minimum distance $d = d(C)$, then we write $[n, k, d]_q$ to show this information. For a linear $[n, k, d]_q$ code C it holds that $k \leq n - d + 1$ (Singleton bound), and C is called *optimal* $[n, k]_q$ code if its minimum weight d achieves the Singleton bound for the minimum weight of $[n, k]_q$ linear codes, i.e., if $d = n - k + 1$. A linear code is *near-optimal* if its minimum distance is at most 1 less than the largest possible value given by the Singleton bound. The *dual* code $C^\perp$ is the orthogonal complement under the standard inner product $(\cdot, \cdot)$, *i.e.* $C^\perp = \{v \in F^n | (v, c) = 0 \text{ for all } c \in C\}$. If $C \subset C^\perp$, then C is called *self-orthogonal*.

We use GAP [11] for all computations in this paper involving the construction of codes from orbit matrices of the design. The obtained codes were analyzed using Magma [2].

2. Orbit matrices of 2-designs

For the construction of self-orthogonal codes we use point orbit matrices of 2-designs introduced in [5].

Definition 2.1. *An $m \times n$ matrix $O = (o_{ij})$ is called a point orbit matrix for parameters (v, b, r, k, λ) and orbit length distributions $(\nu_1, \ldots, \nu_m)$ and $(\beta_1, \ldots, \beta_n)$ if the entries of O satisfy the conditions:*

$$0 \leq o_{ij} \leq \beta_j, 1 \leq i \leq m, 1 \leq j \leq n,$$

$$\sum_{j=1}^{n} o_{ij} = r, 1 \leq i \leq m,$$

$$\sum_{i=1}^{m} \frac{\nu_i}{\beta_j} o_{ij} = k, 1 \leq j \leq n,$$

$$\sum_{j=1}^{n} \frac{\nu_t}{\beta_j} o_{sj} o_{tj} = \lambda \nu_t + \delta_{st}(r - \lambda), 1 \leq s, t \leq m, \tag{2.1}$$

where $\sum\limits_{i=1}^{m} \nu_i = v$, $\sum\limits_{j=1}^{n} \beta_j = b, b = \frac{vr}{k}$.

The submatrix of an orbit matrix corresponding to orbits of size 1 is called the *fixed part of the orbit matrix*, while the submatrix whose rows or columns correspond to orbits of length greater than 1 is called the *nonfixed part of the orbit matrix*.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be 2-(v, b, r, k, λ) design and let G be an automorphism group of $\mathcal{D}$ acting on the set of points with m point orbits denoted by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$, and on the set of blocks with n block orbits denoted by $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$, where $|\mathcal{P}_i| = \nu_i, 1 \leq i \leq m$, and $|\mathcal{B}_j| = \beta_j, 1 \leq j \leq n$. Further, let $\{w_i | 0 \leq i \leq d\}$ be the set of lengths of point and block orbits for G acting on $\mathcal{D}$, where $w_i < w_j$, for $i < j$. Then $m = \sum_{i=0}^{d} m_i$ and $n = \sum_{i=0}^{d} n_i$, where m_i (resp. n_i) is the number of point (resp. block) orbits of length $w_i, i \in \{0, 1, 2, \ldots, d\}$. Let N_j (resp. M_j) be the number of block (resp. point) orbits with size smaller than w_j, that is:

$$N_j = \begin{cases} 0, & j = 0 \\ \sum\limits_{i<j} n_i, & j \in \{1, \ldots, d+1\} \end{cases}, \quad M_j = \begin{cases} 0, & j = 0 \\ \sum\limits_{i<j} m_i, & j \in \{1, \ldots, d+1\} \end{cases}.$$

It follows:

$$\beta_i = w_j, \quad 1 + N_j \leq i \leq N_{j+1}, j \in \{0, \ldots, d\},$$
$$\nu_i = w_j, \quad 1 + M_j \leq i \leq M_{j+1}, j \in \{0, \ldots, d\}.$$

Denote by o_{ij} the number of blocks of $\mathcal{B}_j$ incident with a representative of the point orbit $\mathcal{P}_i$ and denote by γ_{ij} the number of points of $\mathcal{P}_i$ incident with a representative of the block orbit $\mathcal{B}_j$. Then it follows

$$\gamma_{ij} \cdot \beta_j = o_{ij} \cdot \nu_i. \tag{2.2}$$

We call the matrix $O = (o_{ij})$ the point-block orbit matrix, while the matrix $\Gamma = (\gamma_{ij})$ is the block-point orbit matrix of $\mathcal{D}$ for the action of the group G. Note that the matrix O is a point orbit matrix for the parameters (v, b, r, k, λ) and the orbit length distributions $(\nu_1, \ldots, \nu_m)$ and $(\beta_1, \ldots, \beta_n)$. On the other hand, the orbit matrices from Definition 2.1 need not be orbit matrices corresponding to any 2-design.

3. Self-orthogonal codes from orbit matrices of 2-designs

In [5] and [8], constructions of self-orthogonal codes from orbit matrices, and fixed and nonfixed parts of orbit matrices of 2-designs admitting an automorphism group G acting on the set of points and the set of blocks with two orbit lengths, 1 and w, are presented. In this section we present methods for a construction of self-orthogonal codes from point-block orbit matrices of 2-designs, where the lengths of the orbits are not restricted to 1 and w. We use the notation introduced in the previous section and all orbit matrices of designs are point-block orbit matrices.

In the next theorem we study the conditions under which a self-orthogonal code can be constructed from the fixed part of the orbit matrix for an automorphism group $G \leq Aut(D)$ of the 2-design $\mathcal{D}$.

Theorem 3.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, b, r, k, λ) design with an automorphism group G and let O be the corresponding orbit matrix. If $w_0 = 1$ and a prime p divides r, λ and w_h, for all $h \in \{1, 2, \ldots, d\}$ such that $n_h \neq 0$, then the rows of the fixed part of O span a self-orthogonal code of length n_0 over $\mathbb{F}_q$, where $q = p^c$.*

Proof. The fixed part of the orbit matrix $O = (o_{ij})$ is shown in Figure 1.

O		n_0	n_1	n_2	$\ldots$	n_d
		$1 \ 1 \ \ldots \ 1$	$w_1 \ \ldots \ w_1$	$w_2 \ \ldots \ w_2$	$\ldots$	$w_d \ \ldots \ w_d$
m_0	1 $\vdots$ 1					
m_1	w_1 $\vdots$ w_1					
$\vdots$	$\vdots$					
m_d	w_d $\vdots$ w_d					

FIGURE 1. The fixed part of an orbit matrix

Let $\nu_s = \nu_t = 1$, where $1 \leq s \leq t \leq M_1 = m_0$. Then

$$\sum_{j=1}^{n} \frac{\nu_t}{\beta_j} o_{sj} o_{tj} = \sum_{\substack{h=0 \\ n_h \neq 0}}^{d} (\sum_{j=1+N_h}^{N_{h+1}} \frac{1}{w_h} o_{sj} o_{tj}) = \sum_{j=1}^{n_0} o_{sj} o_{tj} + \sum_{\substack{h=1 \\ n_h \neq 0}}^{d} (\sum_{j=1+N_h}^{N_{h+1}} \frac{1}{w_h} o_{sj} o_{tj}),$$

and, using (2.1), it follows

$$\sum_{j=1}^{n_0} o_{sj} o_{tj} = \lambda + \delta_{st}(r - \lambda) - \sum_{\substack{h=1 \\ n_h \neq 0}}^{d} \sum_{j=1+N_h}^{N_{h+1}} \frac{1}{w_h} o_{sj} o_{tj}.$$

Since a point from an orbit of size 1 is incident with none or with all blocks from some block orbit, it follows that $o_{sj} \in \{0, w_h\}$, that is, $o_{sj} o_{tj} \in \{0, w_h^2\}$, for $1 + N_h \leq j \leq N_{h+1}$ such that $n_h \neq 0$.

Thus, we have

$$\sum_{j=1}^{n_0} o_{sj} o_{tj} = \lambda + \delta_{st}(r-\lambda) - \sum_{\substack{h=1 \\ n_h \neq 0}}^{d} w_h z_h,$$

where $z_h = |\{j : o_{sj} o_{tj} = w_h^2\}|$. It follows that the fixed part of the matrix O generates a self-orthogonal code of length n_0 over $\mathbb{F}_q$, where $q = p^c$. □

In the next example, we apply Theorem 3.1 to construct self-orthogonal codes from the fixed part of orbit matrices of a 2-$(1024, 496, 240)$ design.

Example 3.2. *The symmetric* 2-$(1024, 496, 240)$ *design available at* [17] *(see also* [3]*) belongs to the family of Cantor designs. The full automorphism group of this design has order* $2^{35} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31 = 25410822678459187200$*, it is isomorphic to the group* $E_{1024} : O(11, 2)$ *and it has* 73 *conjugacy classes of subgroups isomorphic to* Z_4*. Among them, only a few have fixed orbits and the corresponding orbit length distributions are given in the first column of Table 1, where* d_i *denotes the number of block orbits of length* i*, for* $i \in \{1, 2, 4\}$*. So,* $r = 496$*,* $\lambda = 240$ *and the orbit lengths are 1, 2 or 4. Note that since in addition to the fixed part, there are also parts corresponding to orbits of length 2 and 4, methods described in* [5] *and* [8] *can not be applied. However, the conditions of Theorem 3.1 are satisfied for* $p = 2$ *and from the fixed part of the corresponding orbit matrices we have constructed self-orthogonal binary codes as presented in Table 1. Optimal codes are denoted by* $*$*, and near-optimal codes are denoted by* $**$*.*

Table 1. Self-orthogonal codes from the fixed part of the orbit matrices for Z_4 acting on a 2-$(1024, 496, 240)$ design

(d_1, d_2, d_4)	C	$\lvert\mathrm{Aut}(C)\rvert$	*Weight Distribution*
$(8, 28, 240)$	$[8, 2, 4]^{**}$	$2^7 \cdot 3^2$	$[\langle 0,1\rangle, \langle 4,2\rangle, \langle 8,1\rangle]$
$(16, 120, 192)$	$[16, 2, 8]$	$2^{15} \cdot 3^4 \cdot 5^2 \cdot 7^2$	$[\langle 0,1\rangle, \langle 8,2\rangle, \langle 16,1\rangle]$
$(16, 24, 240)$	$[16, 4, 4]$	$2^{15} \cdot 3^5$	$[\langle 0,1\rangle, \langle 4,4\rangle, \langle 8,6\rangle, \langle 12,4\rangle, \langle 16,1\rangle]$
$(32, 112, 190)$	$[32, 4, 8]$	$2^{31} \cdot 3^9 \cdot 5^4 \cdot 7^4$	$[\langle 0,1\rangle, \langle 8,4\rangle, \langle 16,6\rangle, \langle 24,4\rangle, \langle 32,1\rangle]$
$(32, 112, 190)$	$[32, 4, 16]^{*}$	$2^{30} \cdot 3^9 \cdot 7$	$[\langle 0,1\rangle, \langle 16,14\rangle, \langle 32,1\rangle]$
$(64, 96, 192)$	$[64, 6, 24]$	$2^{56} \cdot 3^{18} \cdot 5$	$[\langle 0,1\rangle, \langle 24,16\rangle, \langle 32,30\rangle, \langle 40,16\rangle, \langle 64,1\rangle]$
$(64, 96, 192)$	$[64, 6, 32]^{*}$	$2^{47} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	$[\langle 0,1\rangle, \langle 32,62\rangle, \langle 64,1\rangle]$
$(128, 64, 192)$	$[128, 8, 56]$	$2^{79} \cdot 3^4 \cdot 5 \cdot 7$	$[\langle 0,1\rangle, \langle 56,64\rangle, \langle 64,126\rangle, \langle 72,64\rangle, \langle 128,1\rangle]$

Using Theorem 3.1 we obtain the following statement.

Corollary 3.3. *Let* $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ *be a* 2-(v, k, λ) *design with an automorphism group* G *and let* O *be the corresponding orbit matrix. If* $w_0 = 1$ *and a prime* p *divides* r, λ *and* w_h*, for all* $h \in \{1, 2, \ldots, d\}$ *such that* $n_h \neq 0$*, then the code spanned by the rows of* O *corresponding to orbits of length* w_0 *is a self-orthogonal code of length* n *over* $\mathbb{F}_q$*, where* $q = p^c$*.*

Proof. Let $\nu_s = \nu_t = w_0 = 1$, where $1 \le s \le t \le M_1$ (Figure 2), and $O = (o_{ij})$.

O		n_0	n_1	n_2	$\dots$	n_d
		$1 \;\dots\; 1$	$w_1 \;\dots\; w_1$	$w_2 \;\dots\; w_2$	$\dots$	$w_d \;\dots\; w_d$
m_0	1 $\vdots$ 1					
m_1	w_1 $\vdots$ w_1					
m_2	w_2 $\vdots$ w_2					
$\vdots$	$\vdots$				$\ddots$	
m_d	w_d $\vdots$ w_d					

FIGURE 2. A submatrix of the orbit matrix O related to the Corollary 3.3

Then

$$\sum_{j=1}^{n} o_{sj}o_{tj} = \sum_{\substack{h=0\\ n_h\neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} o_{sj}o_{tj}\Big) = \sum_{j=1+N_0}^{N_1} o_{sj}o_{tj} + \sum_{\substack{h=1\\ n_h\neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} o_{sj}o_{tj}\Big)$$

From (2.2), we get

$$\begin{aligned}
\sum_{j=1}^{n} o_{sj}o_{tj} &= \sum_{j=1}^{N_1} o_{sj}o_{tj} + \sum_{\substack{h=1\\ n_h\neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} \gamma_{sj}\frac{\beta_j}{\nu_s}\gamma_{tj}\frac{\beta_j}{\nu_t}\Big)\\
&= \sum_{j=1}^{N_1} o_{sj}o_{tj} + \sum_{\substack{h=1\\ n_h\neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} \gamma_{sj}\frac{w_h}{w_0}\gamma_{tj}\frac{w_h}{w_0}\Big)\\
&= \sum_{j=1}^{N_1} o_{sj}o_{tj} + \sum_{\substack{h=1\\ n_h\neq 0}}^{d} (pb_h)^2 \sum_{j=1+N_h}^{N_{h+1}} \gamma_{sj}\gamma_{tj},
\end{aligned}$$

where $pb_h = w_h$ when $h \in \{1,\dots,d\}$ and $n_h \neq 0$.

Since p divides r, λ and w_h, for all $h \in \{1,2,\dots,d\}$ such that $n_h \neq 0$ it follows from Theorem 3.1 that $\sum_{j=1}^{N_1} o_{sj}o_{tj} \equiv 0 \pmod p$. Thus, the linear code spanned by the rows corresponding to orbits of length 1 is a self-orthogonal code of length n over $\mathbb{F}_q$, where $q = p^c$.

□

Theorem 3.1 gives us the conditions under which a self-orthogonal code can be constructed from a fixed part of an orbit matrix. The next theorem gives the construction of a self-orthogonal code also from the other submatrices of an orbit matrix, but some additional properties are required.

Theorem 3.4. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, k, λ) design with an automorphism group G and let O be the corresponding orbit matrix such that $pw_h | w_\ell$ for $h < \ell$ ($pw_\ell | w_h$ for $h > \ell$), where $n_h \neq 0$ and $0 \leq \ell \leq d$. If a prime p divides $r - \lambda$ then the code spanned by the rows of the submatrix of O corresponding to orbits of length w_ℓ is a self-orthogonal code of length n_ℓ over $\mathbb{F}_q$, where $q = p^c$.*

Proof. Let $\nu_s = \nu_t = w_\ell$, where $1 + M_\ell \leq s \leq t \leq M_{\ell+1}$ (Figure 3), and $O = (o_{ij})$.

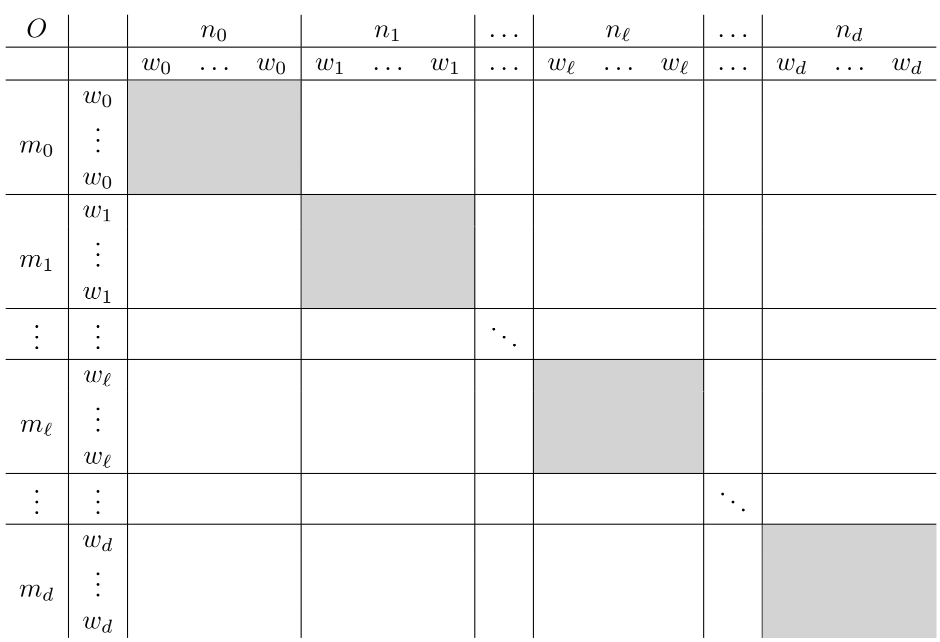

FIGURE 3. Submatrices of the orbit matrix O related to the Theorem 3.4

Then

$$\begin{aligned}\sum_{j=1}^{n}\frac{\nu_t}{\beta_j}o_{sj}o_{tj} &= \sum_{\substack{h=0\\ n_h\neq 0}}^{d}\Big(\sum_{j=1+N_h}^{N_{h+1}}\frac{w_\ell}{w_h}o_{sj}o_{tj}\Big)\\ &= \sum_{\substack{h=0\\ n_h\neq 0}}^{\ell-1}\Big(\sum_{j=1+N_h}^{N_{h+1}}\frac{w_\ell}{w_h}o_{sj}o_{tj}\Big)+\sum_{j=1+N_\ell}^{N_{\ell+1}}\frac{w_\ell}{w_\ell}o_{sj}o_{tj}\\ &\quad+\sum_{\substack{h=\ell+1\\ n_h\neq 0}}^{d}\Big(\sum_{j=1+N_h}^{N_{h+1}}\frac{w_\ell}{w_h}o_{sj}o_{tj}\Big).\end{aligned}$$

From (2.1) follows

$$\begin{aligned}\sum_{j=1+N_\ell}^{N_{\ell+1}}o_{sj}o_{tj} =&\lambda w_\ell+\delta_{st}(r-\lambda)-\sum_{\substack{h=0\\ n_h\neq 0}}^{\ell-1}\Big(\sum_{j=1+N_h}^{N_{h+1}}\frac{w_\ell}{w_h}o_{sj}o_{tj}\Big)\\ &-\sum_{\substack{h=\ell+1\\ n_h\neq 0}}^{d}\Big(\sum_{j=1+N_h}^{N_{h+1}}\frac{w_\ell}{w_h}o_{sj}o_{tj}\Big).\end{aligned}$$

From (2.2) and $pw_h|w_\ell$ for all $h\in\{0,\dots,\ell-1\}$ such that $n_h\neq 0$, we get

$$\begin{aligned}\sum_{j=1+N_\ell}^{N_{\ell+1}}o_{sj}o_{tj} =&\lambda w_\ell+\delta_{st}(r-\lambda)-\sum_{\substack{h=0\\ n_h\neq 0}}^{\ell-1}pc_h\sum_{j=1+N_h}^{N_{h+1}}o_{sj}o_{tj}\\ &-\sum_{\substack{h=\ell+1\\ n_h\neq 0}}^{d}\sum_{j=1+N_h}^{N_{h+1}}\frac{w_\ell}{w_h}\gamma_{sj}\frac{w_h}{w_\ell}\gamma_{tj}\frac{w_h}{w_\ell},\end{aligned}$$

where $pc_h=\frac{w_\ell}{w_h}$ when $0\leq h<\ell$ and $n_h\neq 0$. Since $pw_\ell|w_h$ for all $h\in\{\ell+1,\dots,d\}$ such that $n_h\neq 0$, it holds

$$\begin{aligned}\sum_{j=1+N_\ell}^{N_{\ell+1}}o_{sj}o_{tj} =&\lambda w_\ell+\delta_{st}(r-\lambda)-\sum_{\substack{h=0\\ n_h\neq 0}}^{\ell-1}pc_h\sum_{j=1+N_h}^{N_{h+1}}o_{sj}o_{tj}\\ &-\sum_{\substack{h=\ell+1\\ n_h\neq 0}}^{d}pb_h\sum_{j=1+N_h}^{N_{h+1}}\gamma_{sj}\gamma_{tj},\end{aligned}$$

where $pb_h=\frac{w_h}{w_\ell}$ when $h\in\{\ell+1,\dots,d\}$ and $n_h\neq 0$.

Further, since p divides w_ℓ and $r - \lambda$, it follows that $\sum_{j=1+N_\ell}^{N_{\ell+1}} o_{sj} o_{tj} \equiv 0 \pmod{p}$. Thus, the linear code spanned by the rows corresponding to orbits of length w_ℓ is a self-orthogonal code of length $N_{\ell+1} - N_\ell = n_\ell$ over $\mathbb{F}_q$, where $q = p^c$. □

In the next example, we use Theorem 3.4 to construct self-orthogonal codes from submatrices of orbit matrices of Steiner 2-$(45,5,1)$ designs.

Example 3.5. *Recently, the existence of* 35 *new* 2-$(45,5,1)$ *designs has been established in* [10]. *Incidence matrices of* 30 *previously known* 2-$(45,5,1)$ *designs are available at* [14] *(see also* [4], [6], [13], [15]*). We used the orbit matrices of all known* 2-$(45,5,1)$ *designs that can be obtained for an action of a group of order* 4. *From these we constructed self-orthogonal binary codes. Since* $p = 2$, $r - \lambda = 10$, *and the orbit lengths can be 1, 2 or 4, the conditions of Theorem 3.4 are satisfied.*

In Tables 2 and 3, we present self-orthogonal binary codes from orbits of length 4 from orbit matrices for the action of group Z_4 *and* E_4, *respectively. In the first column of the table, the corresponding distributions of orbit lengths are given, where* d_i *denotes the number of block orbits of length* i, *for* $i \in \{1,2,4\}$. *As in the previous example, in these orbit matrices, in addition to the part corresponding to orbits of length 4, there is also a part corresponding to orbits of length 1 and 2, so we can not obtain these results with the constructions known so far. Optimal codes are denoted by* $*$, *and near-optimal codes are denoted by* $**$.

Table 2. Codes from the parts corresponding to the orbits of length 4 from the orbit matrices for Z_4 acting on 2-$(45, 5, 1)$ designs

(d_1, d_2, d_4)	C	$\lvert\mathrm{Aut}(C)\rvert$	*Weight Distribution*
$(3, 8, 20)$	$[20, 5, 8]^{**}$	2^{12}	$[\langle 0,1\rangle,\langle 8,14\rangle,\langle 12,16\rangle,\langle 16,1\rangle]$
$(3, 8, 20)$	$[20, 6, 8]^{*}$	2^{9}	$[\langle 0,1\rangle,\langle 8,32\rangle,\langle 12,28\rangle,\langle 16,3\rangle]$
$(3, 4, 22)$	$[22, 6, 6]$	2^{4}	$[\langle 0,1\rangle,\langle 6,1\rangle,\langle 8,9\rangle,\langle 10,19\rangle,\langle 12,22\rangle,$ $\langle 14,11\rangle,\langle 18,1\rangle]$
$(3, 4, 22)$	$[22, 7, 4]$	2^{4}	$[\langle 0,1\rangle,\langle 4,1\rangle,\langle 6,1\rangle,\langle 8,19\rangle,\langle 10,39\rangle,$ $\langle 12,43\rangle,\ \langle 14,23\rangle,\langle 18,1\rangle]$
$(3, 4, 22)$	$[22, 7, 4]$	2^{5}	$[\langle 0,1\rangle,\langle 4,1\rangle,\langle 6,1\rangle,\langle 8,23\rangle,\langle 10,39\rangle,$ $\langle 12,35\rangle,\langle 14,23\rangle,\langle 16,4\rangle,\langle 18,1\rangle]$
$(3, 4, 22)$	$[22, 7, 4]$	2^{6}	$[\langle 0,1\rangle,\langle 4,2\rangle,\langle 8,18\rangle,\langle 10,40\rangle,$ $\langle 12,42\rangle,\langle 14,24\rangle,\langle 16,1\rangle]$
$(3, 4, 22)$	$[22, 7, 4]$	2^{6}	$[\langle 0,1\rangle,\langle 4,1\rangle,\langle 6,3\rangle,\langle 8,21\rangle,\langle 10,35\rangle,$ $\langle 12,39\rangle,\langle 14,25\rangle,\langle 16,2\rangle,\langle 18,1\rangle]$
$(3, 4, 22)$	$[22, 7, 4]$	2^{8}	$[\langle 0,1\rangle,\langle 4,2\rangle,\langle 6,2\rangle,\langle 8,18\rangle,\langle 10,38\rangle,$ $\langle 12,42\rangle,\langle 14,22\rangle,\langle 16,1\rangle,\langle 18,2\rangle]$
$(1, 5, 22)$	$[22, 8, 4]$	$2^{4}\cdot 3$	$[\langle 0,1\rangle,\langle 4,1\rangle,\langle 8,75\rangle,\langle 12,163\rangle,\ \langle 16,16\rangle]$
$(1, 5, 22)$	$[22, 8, 4]$	2^{9}	$[\langle 0,1\rangle,\langle 4,2\rangle,\langle 8,76\rangle,\langle 12,158\rangle,\ \langle 16,19\rangle]$
$(3, 4, 22)$	$[22, 8, 4]$	2^{9}	$[\langle 0,1\rangle,\langle 4,4\rangle,\langle 6,2\rangle,\langle 8,46\rangle,\langle 10,78\rangle,$ $\langle 12,68\rangle,\langle 14,46\rangle,\langle 16,9\rangle,\langle 18,2\rangle]$
$(3, 4, 22)$	$[22, 8, 4]$	2^{10}	$[\langle 0,1\rangle,\langle 4,4\rangle,\langle 6,4\rangle,\langle 8,42\rangle,\langle 10,76\rangle,$ $\langle 12,76\rangle,\langle 14,44\rangle,\langle 16,5\rangle,\langle 18,4\rangle]$
$(3, 4, 22)$	$[22, 8, 6]$	4	$[\langle 0,1\rangle,\langle 6,8\rangle,\langle 8,38\rangle,\langle 10,80\rangle,\langle 12,80\rangle,$ $\langle 14,40\rangle,\langle 16,9\rangle]$
$(3, 4, 22)$	$[22, 8, 6]$	4	$[\langle 0,1\rangle,\langle 6,10\rangle,\langle 8,36\rangle,\langle 10,76\rangle,\langle 12,84\rangle,$ $\langle 14,42\rangle,\langle 16,7\rangle]$
$(3, 4, 22)$	$[22, 8, 6]$	2^{6}	$[\langle 0,1\rangle,\langle 6,8\rangle,\langle 8,42\rangle,\langle 10,80\rangle,\langle 12,72\rangle,$ $\langle 14,40\rangle,\langle 16,13\rangle]$
$(1, 5, 22)$	$[22, 8, 8]^{*}$	2^{4}	$[\langle 0,1\rangle,\langle 8,78\rangle,\langle 12,160\rangle,\langle 16,17\rangle]$
$(1, 5, 22)$	$[22, 8, 8]^{*}$	$2^{6}\cdot 3$	$[\langle 0,1\rangle,\langle 8,86\rangle,\langle 12,144\rangle,\langle 16,25\rangle]$
$(1, 5, 22)$	$[22, 8, 8]^{*}$	$2^{11}\cdot 3$	$[\langle 0,1\rangle,\langle 8,90\rangle,\langle 12,136\rangle,\langle 16,29\rangle]$
$(3, 4, 22)$	$[22, 10, 4]$	2^{12}	$[\langle 0,1\rangle,\langle 4,4\rangle,\langle 6,32\rangle,\langle 8,158\rangle,\langle 10,320\rangle,$ $\langle 12,308\rangle,\langle 14,160\rangle,\langle 16,41\rangle]$
$(1, 5, 22)$	$[22, 10, 4]$	$2^{12}\cdot 3^{2}$	$[\langle 0,1\rangle,\langle 4,4\rangle,\langle 8,318\rangle,\langle 12,628\rangle,\langle 16,73\rangle]$
$(1, 5, 22)$	$[22, 10, 4]$	$2^{12}\cdot 3^{2}$	$[\langle 0,1\rangle,\langle 4,8\rangle,\langle 8,306\rangle,\langle 12,640\rangle,\langle 16,69\rangle]$
$(3, 4, 22)$	$[22, 10, 4]$	$2^{12}\cdot 3^{2}$	$[\langle 0,1\rangle,\langle 4,8\rangle,\langle 6,32\rangle,\langle 8,146\rangle,\langle 10,320\rangle,$ $\langle 12,320\rangle,\langle 14,160\rangle,\langle 16,37\rangle]$
$(3, 4, 22)$	$[22, 10, 6]$	$3\cdot 5\cdot 2^{8}$	$[\langle 0,1\rangle,\langle 6,32\rangle,\langle 8,170\rangle,\langle 10,320\rangle,$ $\langle 12,296\rangle,\langle 14,160\rangle,\langle 16,45\rangle]$
$(1, 5, 22)$	$[22, 10, 8]^{*}$	$2^{8}\cdot 3^{2}\cdot 5\cdot 7\cdot 11$	$[\langle 0,1\rangle,\langle 8,330\rangle,\langle 12,616\rangle,\langle 16,77\rangle]$

TABLE 3. Codes from the parts corresponding to the orbits of length 4 from the orbit matrices for $Z_2 \times Z_2$ acting on 2-$(45, 5, 1)$ designs

(d_1, d_2, d_4)	C	$\lvert\mathrm{Aut}(C)\rvert$	*Weight Distribution*
$(5, 17, 15)$	$[15, 3, 8]^*$	$2^7 \cdot 3^5$	$[\langle 0,1\rangle, \langle 8,6\rangle, \langle 12,1\rangle]$
$(5, 17, 15)$	$[15, 4, 4]^*$	$2^9 \cdot 3$	$[\langle 0,1\rangle, \langle 4,1\rangle, \langle 8,13\rangle, \langle 12,1\rangle]$
$(3, 16, 16)$	$[16, 4, 8]^*$	$2^{10} \cdot 3$	$[\langle 0,1\rangle, \langle 8,13\rangle, \langle 12,2\rangle]$
$(3, 16, 16)$	$[16, 5, 4]$	2^9	$[\langle 0,1\rangle, \langle 4,2\rangle, \langle 8,25\rangle, \langle 12,4\rangle]$
$(3, 16, 16)$	$[16, 5, 4]$	2^{11}	$[\langle 0,1\rangle, \langle 4,3\rangle, \langle 8,23\rangle, \langle 12,5\rangle]$
$(3, 16, 16)$	$[16, 5, 4]$	$2^8 \cdot 3^2$	$[\langle 0,1\rangle, \langle 4,1\rangle, \langle 8,27\rangle, \langle 12,3\rangle]$
$(5, 9, 19)$	$[19, 5, 6]$	$2^4 \cdot 3$	$[\langle 0,1\rangle, \langle 6,1\rangle, \langle 8,9\rangle, \langle 10,14\rangle, \langle 12,6\rangle, \langle 14,1\rangle]$
$(5, 9, 19)$	$[19, 5, 8]^*$	$2^7 \cdot 3^2$	$[\langle 0,1\rangle, \langle 8,9\rangle, \langle 10,16\rangle, \langle 12,6\rangle]$
$(5, 9, 19)$	$[19, 6, 4]$	2^6	$[\langle 0,1\rangle, \langle 4,1\rangle, \langle 6,2\rangle, \langle 8,19\rangle, \langle 10,28\rangle, \langle 12,11\rangle, \langle 14,2\rangle]$
$(5, 9, 19)$	$[19, 6, 4]$	2^{10}	$[\langle 0,1\rangle, \langle 4,2\rangle, \langle 8,18\rangle, \langle 10,32\rangle, \langle 12,10\rangle, \langle 16,1\rangle]$
$(5, 9, 19)$	$[19, 6, 4]$	$2^9 \cdot 3$	$[\langle 0,1\rangle, \langle 4,3\rangle, \langle 8,15\rangle, \langle 10,32\rangle, \langle 12,13\rangle]$

The next corollary is a consequence of Theorem 3.4.

Corollary 3.6. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, k, λ) design with an automorphism group G and let O be the corresponding orbit matrix such that $pw_h | w_\ell$ for $h < \ell$ ($pw_\ell | w_h$ for $h > \ell$), where $n_h \neq 0$ and $0 \leq \ell \leq d$. Let A_ℓ be the submatrix of O such that the rows of A_ℓ correspond to orbits of length w_ℓ and the columns of A_ℓ correspond to orbits of length greater than or equal to w_ℓ. If a prime p divides $r - \lambda$, then the code spanned by the rows of the submatrix A_ℓ is a self-orthogonal code of length $n - N_\ell$ over $\mathbb{F}_q$, where $q = p^c$.*

Proof. Let $\nu_s = \nu_t = w_\ell$, where $1 + M_\ell \leq s \leq t \leq M_{\ell+1}$ (Figure 4), and $O = (o_{ij})$.

Then

$$\sum_{j=1+N_\ell}^{n} o_{sj} o_{tj} = \sum_{\substack{h=\ell \\ n_h \neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} o_{sj} o_{tj} \Big) = \sum_{j=1+N_\ell}^{N_{\ell+1}} o_{sj} o_{tj} + \sum_{\substack{h=\ell+1 \\ n_h \neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} o_{sj} o_{tj} \Big)$$

From (2.2) we get

$$\begin{aligned} \sum_{j=1+N_\ell}^{n} o_{sj} o_{tj} &= \sum_{j=1+N_\ell}^{N_{\ell+1}} o_{sj} o_{tj} + \sum_{\substack{h=\ell+1 \\ n_h \neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} \gamma_{sj} \frac{\beta_j}{\nu_s} \gamma_{tj} \frac{\beta_j}{\nu_t} \Big) \\ &= \sum_{j=1+N_\ell}^{N_{\ell+1}} o_{sj} o_{tj} + \sum_{\substack{h=\ell+1 \\ n_h \neq 0}}^{d} \Big(\sum_{j=1+N_h}^{N_{h+1}} \gamma_{sj} \frac{w_h}{w_\ell} \gamma_{tj} \frac{w_h}{w_\ell} \Big). \end{aligned}$$

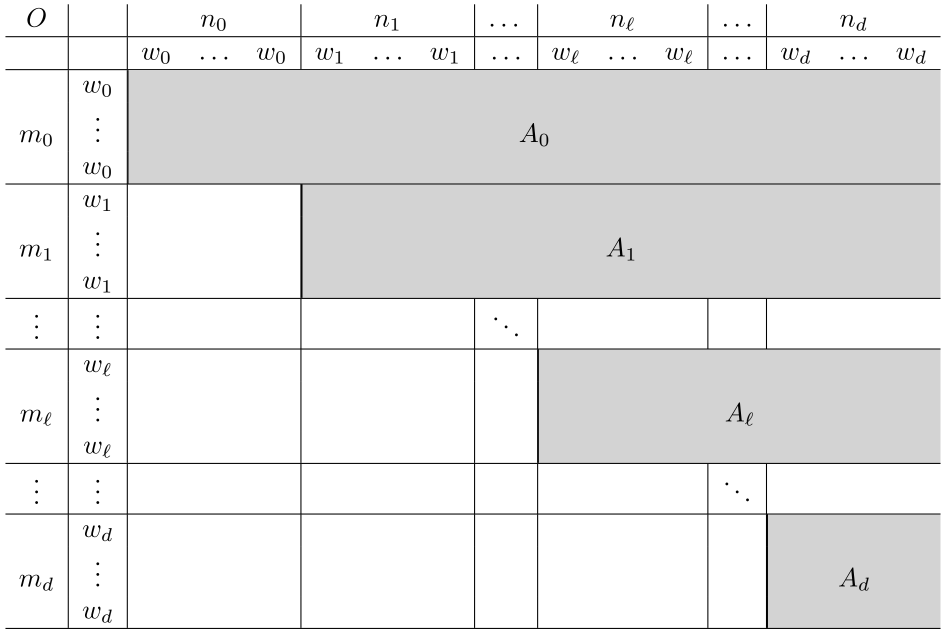

FIGURE 4. Submatrices of the orbit matrix O related to the Corollary 3.6

Since $pw_\ell | w_h$ for all $h \in \{\ell+1, \dots, d\}$ such that $n_h \neq 0$ we have

$$\sum_{j=1+N_\ell}^{n} o_{sj}o_{tj} = \sum_{j=1+N_\ell}^{N_{\ell+1}} o_{sj}o_{tj} + \sum_{\substack{h=\ell+1 \\ n_h \neq 0}}^{d} (pb_h)^2 \sum_{j=1+N_h}^{N_{h+1}} \gamma_{sj}\gamma_{tj},$$

where $pb_h = \frac{w_h}{w_\ell}$ when $h \in \{\ell+1, \dots, d\}$ and $n_h \neq 0$.

From Thereom 3.4 it follows that $\sum_{j=1+N_\ell}^{n} o_{sj}o_{tj} \equiv 0 \pmod p$, and the linear code spanned by the rows of the submatrix A_ℓ is a self-orthogonal code of length $n - N_\ell$ over $\mathbb{F}_q$, where $q = p^c$.

□

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Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `mmaksimovic@math.uniri.hr`

Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `sanjar@math.uniri.hr`