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ASYMPTOTIC ESTIMATE ON THE DISTANCE ENERGY OF LATTICES

ZHIPENG LU AND XIANCHANG MENG

ABSTRACT. Since the well-known breakthrough of L. Guth and N. Katz on the Erdős distinct distances problem in the plane, it aroused mainstream interest by their method and the Elekes–Sharir framework. In short, they study the second moment in the framework. One may wonder if higher moments would be more efficient. In this paper, using number-theoretic methods, we show that any higher moment fails the expectation. We also show that the second moment gives an optimal estimate in higher dimensions. Moreover, we prove the mean second moment attains the maximum for the hexagonal lattice in $\mathbb{R}^2$, which is a parallel result on the distinct distances problem for lattices.

1. INTRODUCTION

The Erdős conjecture on distinct distances in the Euclidean plane $\mathbb{R}^2$ says, $d(P) := |\{d(p,q) \mid p,q \in P\}| \gtrsim \frac{|P|}{\sqrt{\log|P|}}$ for any finite set $P \subset \mathbb{R}^2$, where $d(,)$ is the Euclidean distance and "$\gtrsim x$" always means "$\geq Cx$" for some absolute constant $C > 0$. In Guth–Katz [4], the nearly optimal bound $d(P) \gtrsim \frac{|P|}{\log|P|}$ was established. The authors used a group theoretic framework, called the Elekes–Sharir framework, to reduce the problem of enumerating distinct distances to that of estimating line-line incidences in $\mathbb{R}^3$.

The essential object studied in [4] is a type of energy, which they call *distance quadruples*, i.e. $Q(P) =: \{(p_1, q_1, p_2, p_2) \in P^4 \mid d(p_1, q_1) = d(p_2, q_2)\}$. We call $|Q(P)|$ the *distance energy* of P, denoted by $E_2(P)$. Moreover, we can define $E_k(P) := |\{(p_1, q_1, \ldots, p_k, q_k) \in P^{2k} \mid d(p_1, q_1) = \cdots = d(p_k, q_k)\}|$, and call it the kth distance energy of P. These energies count multiplicities of pairs with common distance and may be compared with a

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different notion called "Riesz energy" in geometric measure theory (see [5] for instance).

In this paper, we consider higher distance energy $E_k(P)$ for $k \geq 3$ and investigate higher moments in the Elekes–Sharir framework with the expectation that it might be more efficient than just estimating $E_2(P)$ as in [4]. Due to technical reasons, we only consider the example of square grids, which yet shows that the expectation is in vain for the Euclidean plane, see Theorem 3.4 below. Moreover, we also study distance energies in $\mathbb{R}^n, n \geq 3$ for the square grid example and show that the second moment gives the truly optimal estimate for the distinct distances problem in higher dimensions, see Corollary 4.3.

Focusing on lattices, Conway and Sloane [2] determined in $\mathbb{R}^n$ $(n \geq 3)$ the lattices with minimal distinct distances, and claimed that in $\mathbb{R}^2$ the lattice should be the hexagonal lattice, which was then affirmed by Moree and Osburn [10]. Following this line, we show that among all lattices in $\mathbb{R}^2$, the hexagonal lattice also attains the maximal (mean) second distance energy asymptotically, see Theorem 5.4. Our estimate relies on Kühnlein's criterion (see [9]) of arithmetic lattices and a key result on the counting values of integral quadratic form by Müller [11]. However, we do not have adequate tools to deal with higher distance energies but to relate them to the minimum of higher moment versions of Epstein zeta functions, which count values of integral quadratic forms. It was shown that the Epstein zeta function attains the minimum only for equivalent forms corresponding to the hexagonal lattices (see Cassels [1] for instance). We wonder if this still holds for the higher moment version of Epstein zeta functions, as conjectured at the end of section 5.

Moreover in the appendix, relying on a "factor-out" tip for summations, we asymptotically estimate the true (not mean) distance energy of square squids, compensating the rough estimate by incidence geometry in [4].

2. Higher moments in the Elekes–Sharir framework

We may define distance energy in any general metric space (M, d). Let $P \subset M$ be a set of N points and $d(P)$ the number of distinct distances between points of P. For each distance $d_i, i = 1, \ldots, d(P)$, let n_i be the number of pairs of points from P at distance d_i. Clearly $\sum_{i=1}^{d(P)} n_i = 2\binom{N}{2} = N(N-1)$. Then we use Hölder's inequality to get the following

Lemma 2.1. *For any positive integer $k \geq 2$,*

$$d(P) \geq \frac{(N^2 - N)^{\frac{k}{k-1}}}{(\sum_{i=1}^{d(P)} n_i^k)^{\frac{1}{k-1}}}.$$

Proof. By Hölder's inequality for $\frac{k-1}{k} + \frac{1}{k} = 1$,

$$N^2 - N = \sum_{i=1}^{d(P)} n_i \le \left(\sum_{i=1}^{d(P)} 1^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} \left(\sum_{i=1}^{d(P)} n_i^k \right)^{1/k} = d(P)^{\frac{k-1}{k}} \left(\sum_{i=1}^{d(P)} n_i^k \right)^{1/k}.$$

By rearranging we get the desired inequality. □

By our definition, $E_k(P) = \sum_{i=1}^{d(P)} n_i^k$. In order to prove Erdős' conjecture in this setting, we need to show

$$E_k(P) \lesssim N^{k+1} (\log N)^{\frac{k-1}{2}}, \forall P \subset \mathbb{R}^2 \text{ with } |P| = N, \tag{2.1}$$

at least for some $k \ge 2$. Guth and Katz [4] already showed that this is not true for $k = 2$. Actually, they proved $E_2(P) \lesssim N^3 \log N$. They also calculated the example where P is a square grid with N points, $E_k(P) \gtrsim N^3 \log N$, see appendix of [4]. Thus, to verify or refute (2.1), we need to estimate $E_k(P)$ at least for the square grid example.

3. Rough estimate on higher distance energy of square grids

In this section, we estimate the higher distance energies $E_k(P)$, specifically, by calculating the Dirichlet series that encodes the number of representations of integers as a sum of squares, which is derived from higher distance energies $E_k(P)$. Note that in the appendix of [4], $E_2(P)$ was estimated by counting line-line incidences in $\mathbb{R}^3$.

Let $P = [\sqrt{N}] \times [\sqrt{N}]$ be the square grid of size N, where $[x]$ denotes the set of integers ranging from 1 to $\lfloor x \rfloor$. Then a piece of energy in $E_k(P)$ is afforded by $a_1^2 + b_2^2 = \cdots = a_k^2 + b_k^2$ for some $a_i, b_i \in [\sqrt{N}], i = 1, \ldots, k$, and some $p_i, q_i \in P$ such that $p_i - q_i = (\pm a_i, \pm b_i)$. Note that for $a_i, b_i \le \frac{\sqrt{N}}{2}$, the number of such pairs (p_i, q_i) is $\gtrsim \sqrt{N} \cdot \sqrt{N} = N$. Denoting $r(n) := |\{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 = n\}|$, we get the rough estimate written as

$$E_k(P) \gtrsim N^k \sum_{n \le \frac{N}{2}} r(n)^k. \tag{3.1}$$

Then the k-moment of the above sum has the following estimate:

Proposition 3.1. *For any positive integer k and $x \in \mathbb{R}_+$, we have the following asymptotics*

$$\sum_{n \le x} r(n)^k \sim x P_{2^{k-1}-1}(\log x),$$

where $P_{2^{k-1}-1}$ is a polynomial of degree $2^{k-1} - 1$.

Note that more precisely for $k = 2$,

$$\sum_{n \le x} r^2(n) \sim 4x \log x + O(x) \tag{3.2}$$

(see Wilson [15]). The general result of Proposition 3.1 seems classical, but we will provide a proof as a detour of consequence by the following two results.

Lemma 3.2 ((7.20) of [15]). *For any positive integer k, there is the following expression for the Dirichlet series:*

$$\sum_{n=1} \frac{r(n)^k}{n^s} = 4^k (1-2^{-s})^{2^{k-1}-1} \left(\zeta(s)\eta(s)\right)^{2^{k-1}} \phi(s), \forall \Re(s) > 1,$$

where $\zeta(s)$ is the Riemann zeta function, $\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots$, and $\phi(s) = \prod_p \left(1 + \sum_{\nu=2}^{\infty} a_\nu p^{-\nu s}\right)$ is absolutely convergent for $\Re(s) > \frac{1}{2}$.

To get the average of $r^k(n)$, we rely on the following form of Perron's formula:

Lemma 3.3 (Theorem 1 of Chapter V in Karatsuba [8]). *Assume that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely for $\Re(s) > 1$, $|a_n| \leq A(n)$ for some monotonically increasing function $A(x) > 0$, and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O((\sigma - 1)^{-\alpha}), \alpha > 0,$$

as $\sigma \to 1_+$. Then for any $b_0 \geq b > 1$, $T \geq 1$, and $x = N + \frac{1}{2}$, we have

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^b}{T(b-1)^\alpha}\right) + O\left(\frac{xA(2x)\log x}{T}\right).$$

Proof of Proposition 3.1. First, $r(n)$ is indeed always of order $o(n^\epsilon)$ for any $\epsilon > 0$ (see, for instance, Theorem 338 of Hardy and Wright [6]). Thus the condition of Lemma 3.3 is easily satisfied. Also note that $\eta(s)$ is holomorphic and poles are only on $\zeta(s)$. Then Wilson's calculation of the Dirichlet series of $r(n)^k$ as in Lemma 3.2 shows that, by estimating the residue integral of contour, the order of $\sum_{n \leq x} r(n)^k$ should be $x(\log x)^{2^{k-1}-1}$, whereas T may tend to be larger than any log power due to irrelevance of choices of contours.

□

Proposition 3.1 together with (3.1) immediately implies the following

Theorem 3.4. *For $k \geq 2$ and large positive integer N ($\gg k$), we have the following estimate on the kth distance energy:*

$$E_k([\sqrt{N}] \times [\sqrt{N}]) \gtrsim N^{k+1} (\log N)^{2^{k-1}-1}.$$

The result turns against the expectation of (2.1) by a big log factor. Moreover, if $N^{k+1}(\log N)^{2^{k-1}}$ is the right order for $E_k(P)$ in general, then by Lemma 2.1, in the Elekes–Sharir framework we may get the most efficient bound $d(P) \gtrsim \frac{N}{\log N}$ only when we study the second moment.

4. Distance energy of square grids in higher dimensions

In addition, we notice that the distance energy of square lattices in higher dimensions is optimal. Consider $P = [\sqrt[m]{N}]^m$ the square grid of size N in $\mathbb{R}^m, m \geq 3$, and let $r_m(n) = |\{(x_1, \dots, x_m) \in \mathbb{Z}^m \mid x_1^2 + \cdots + x_m^2 = n\}|$. Then similarly we have

$$c_1 N^2 \sum_{n \leq N^{2/3}} r_3^2(n) \leq E_2(P) \leq c_2 N^2 \sum_{n \leq N^{2/3}} r_3^2(n),$$

for some $c_1, c_2 > 0$. To this end, we introduce a more general result as follows

Lemma 4.1 (Theorem 6.1 of Müller [11]). *Let $q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x}$ be a primitive positive definite integral quadratic form in $m \geq 3$ variables and $r_Q(n) = |\{\mathbf{x} \in \mathbb{Z}^m \mid q(\mathbf{x}) = n\}|$. Then*

$$\sum_{n \leq x} r_Q^2(n) = B x^{m-1} + O\left(x^{(m-1)\frac{4m-5}{4m-3}}\right),$$

for some constant $B > 0$ depending on Q. For $m = 2$,

$$\sum_{n \leq x} r_Q^2(n) = A_Q x \log x + O(x),$$

where

$$A_Q = 12 \frac{A(q)}{q} \prod_{p \mid q} \left(1 + \frac{1}{p}\right)^{-1}, \ q = \det(Q).$$

Here $A(q)$ denotes the multiplicative function defined by $A(p^e) = 2 + (1 - \frac{1}{p})(e-1)$ for odd prime p, and

$$A(2^e) = \begin{cases} 1, \ \text{if } e \leq 1, \\ 2 \ \text{if } e = 2, \\ e-1, \ \text{if } e \geq 3. \end{cases}$$

This immediately implies

Corollary 4.2. *For $P = [\sqrt[m]{N}]^m$ the square grid of size N in $\mathbb{R}^m, m \geq 3$, we have*

$$N^{2+\frac{2m-2}{m}} \lesssim E_2(P) \lesssim N^{2+\frac{2m-2}{m}}.$$

Note that by Legendre's three-square theorem, $n = x^2 + y^2 + z^2 \leq N^{\frac{2}{3}}$ for $n \neq 4^a(8m+7)$, which amount to $cN^{\frac{2}{3}}$ numbers for some $c > 0$, i.e. $d(P) = c' N^{\frac{2}{3}}, c' > 0$, as Erdős noted for the distinct distances conjecture in $\mathbb{R}^3$. For $m \geq 4$, by Lagrange's four-square theorem, each positive integer can be expressed as a sum of m squares, i.e. $d(P) = cN^{\frac{2}{m}}$. Thus for any $m \geq 3$, we can conclude that

Corollary 4.3. *For any $m \geq 3$, the estimate by distance energy of distinct distances in $\mathbb{R}^m$ is optimal, i.e.*

$$d(P) \lesssim \frac{|P|^4}{E_2(P)},$$

for certain examples like $P = [\sqrt[m]{N}]^m$ the square grid of size N.

This seems to indicate that $E_2(P) \lesssim |P|^{2+\frac{2m-2}{m}}$ for any finite set $P \subset \mathbb{R}^m$ so that a proper estimate for the distance energy may suffice to solve the Erdős conjecture in higher dimensions. However, similar to $\mathbb{R}^2$, we believe that an estimate by higher distance energies would not be optimal.

5. The distance energy for general lattices and Epstein zeta functions

In this section, we consider general lattices and compare their distance energy. Due to technical reasons, we only deal with the pointwise distance energy, i.e. $E_{L,k}(N) := |\{(p_1, \ldots, p_k) \in L^k, \|p_1\|^2 = \cdots = \|p_k\|^2 \leq N\}|$ for any lattice $L \subset \mathbb{R}^2, k \in \mathbb{Z}_{\geq 0}$. Let $r_L(n) = |\{p \in L, \|p\|^2 = n\}|$. Then

$$E_{L,k}(N) = \sum_{n \leq N} r_L^k(n). \tag{5.1}$$

We have already seen the estimates of $E_{L,k}(N)$ for the square lattice L in the last section. Note that $E_{L,0}(N)$ counts the distinct distances.

A general lattice $L \subset \mathbb{R}^2$ of covolume 1, after rotation, may be written as $\mathbb{Z}(a, 0) \oplus \mathbb{Z}(b, \frac{1}{a})$ for some $a, b > 0$. To estimate its distance energy, we need to study the value distribution of the quadratic form $Q_L(x, y) = (ax+by)^2 + \frac{1}{a^2}y^2 = a^2\left(x^2 + 2\frac{b}{a}xy + \left(\frac{1}{a^4} + \frac{b^2}{a^2}\right)y^2\right)$. For example, $a = \sqrt{\frac{2}{\sqrt{3}}}, b = \frac{1}{2}\sqrt{\frac{2}{\sqrt{3}}}$ correspond to the hexagonal lattice (which we will always denote by Σ) and the quadratic form $\frac{2}{\sqrt{3}}(x^2 + xy + y^2)$. If $\frac{a}{b}$ or $\frac{1}{a^4} + \frac{b^2}{a^2}$ is irrational, then integer solutions to $Q_L(x, y) = Q_L(x', y')$ would be very few, i.e. have small distance energy.

We will only be concerned about the lattices with Q_L similar to norms of imaginary quadratic number fields, i.e. *arithmetic* lattices, due to the following Kühnlein's criterion:

Lemma 5.1 (Kühnlein [9])**.** *Let $L \subset \mathbb{R}^2$ be a lattice. Then L is arithmetic if and only if there are at least* 3 *pairwise linearly independent vectors in L which have the same length.*

This immediately implies

Corollary 5.2. *The number of distinct distances in a nonarithmetic lattice grid (square or circular) of N points is $\gtrsim N$ and its distance energy is $O(N^3)$.*

Remark 5.3: For L arithmetic, there is always $E_{L,0}(N) \lesssim \frac{N}{\sqrt{\log N}}$ like the square grids, see Moree and Osburn [10] for details. Moreover, they proved that in $\mathbb{R}^2$ the hexagonal lattice attains the minimal number of distinct distances, i.e. the minimum of $E_{L,0}(N)$ for N large.

Now that arithmetic lattices are nothing but submodules of rings of integers of imaginary quadratic fields, it suffices to consider the norms of those rings. For any negative square-free integer D, if $D \equiv 1 \mod 4$, the ring of integers is $\mathcal{O}_D = \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$ with *discriminant* D; otherwise, $\mathcal{O}_D = \mathbb{Z}[\sqrt{D}]$ with discriminant $4D$. Hence we define $Q_D(x, y)$ by

$$(5.2) \qquad \begin{cases} (x + \frac{1+\sqrt{D}}{2}y)(x + \frac{1-\sqrt{D}}{2}y) = x^2 + xy + \frac{1-D}{4}y^2, D \equiv 1 \text{ mod } 4; \\ (x + \sqrt{D}y)(x - \sqrt{D}y) = x^2 - Dy^2, \text{ otherwise.} \end{cases}$$

Note that the discriminant is just that of Q_D and Q_D are all positive definite. For example, if $D = -1$, it is the Gaussian ring $\mathbb{Z}[i]$ with norm $Q_{-1}(x, y) = x^2+y^2$; if $D = -3$, it is the Eisenstein ring $\mathbb{Z}\left[\frac{1+\sqrt{3}i}{2}\right]$ with norm $Q_{-3}(x, y) = x^2 + xy + y^2$. To have covolume 1, the lattices need to be scaled by $S_D := \sqrt{2}(-D)^{-\frac{1}{4}}$ or $(-D)^{-\frac{1}{4}}$, which was already seen in the case of the hexagonal lattice. This is to ensure that there are always $\sim \pi N$ lattice points in a disc of radius $\sqrt{N}$.

For arithmetic lattices, we may write $r_D(n)$ for $r_L(n)$. Note that $r_{-1}(n) = r(n)$ and $r_{-3}(n) = |\{(x, y) \in \mathbb{Z}^2 \mid x^2 + xy + y^2 = n\}|$ are the only two cases of counting integral points on circles, while the others are on ellipses. Then we also write

$$(5.3) \qquad E_{D,k}(N) = \sum_{n \leq N/S_D^2} r_D^k(n).$$

Then we can use Lemma 4.1 to give the asymptotics of $E_{D,2}(N)$. By calculation we see that

$$(5.4) \qquad E_{-3,2}(N) = 3\sqrt{3}N \log N + O(N),$$

which is larger than $E_{-1,2}(N) = 4N \log N + O(N)$ as we have seen from (3.2). Note that the 2×2 matrices as of Lemma 4.1 are

$$\begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-D}{2} \end{pmatrix}, \text{ if } D \equiv 1 \mod 4, \\ \\ \begin{pmatrix} 2 & 0 \\ 0 & -2D \end{pmatrix}, \text{ otherwise.} \end{cases}$$

By more careful calculation of the coefficients of the main terms, we see the following

Theorem 5.4. *Let D be any square-free negative integer and N be large. Then $E_{D,2}(N) < E_{-3,2}(N)$ for $D \equiv 1 \mod 4$, and $E_{D,2}(N) < E_{-1,2}(N)$*

otherwise. Among all lattices L in $\mathbb{R}^2$, the pointwise distance energy $E_{L,2}(N)$ attains the maximum only when L is the hexagonal lattice.

One may also be interested in higher pointwise energies $E_{D,k}(N)$ for $k \geq 3$. Explicit formulas of $r_D(n)$ may be found in Huard, Kaplan and Williams [7] or Sun and Williams [14], but estimating $E_{D,k}(N)$ from those formulae is hardly possible.

On the other hand, in general $r_Q(n) = |\{(x,y) \in \mathbb{Z}^2 \mid Q(x,y) = n\}|$ is used to define the Epstein zeta function:

$$Z_Q(s) = \sum_{0 \neq x,y \in \mathbb{Z}} \frac{1}{Q(m,n)^s} = \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^s} \tag{5.5}$$

which converges for $\Re s > 1$. Moreover, it can be analytically continued to the whole complex plane with a simple pole at $s = 1$ and satisfies the functional equation ($D = \mathrm{disc}(Q)$)

$$\left(\frac{\sqrt{D}}{2\pi}\right)^s \Gamma(s) Z_Q(s) = \left(\frac{\sqrt{D}}{2\pi}\right)^{1-s} \Gamma(1-s) Z_Q(1-s), \tag{5.6}$$

see for instance Zhang and Williams [14]. There is also a closed formula by Chowla and Selberg (see [13]), which states for $Q(x,y) = ax^2 + bxy + cy^2, D = b^2 - 4ac$,

$$\begin{aligned} Z_Q(s) &= a^{-s}\zeta(2s) + a^{-s}\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)l^{1-2s} + R_Q(s), \\ R_Q(s) &= \frac{4a^{-s}l^{-s+\frac{1}{2}}}{\pi^{-s}\Gamma(s)} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} (\sum_{d|n} d^{1-2s}) K_{s-\frac{1}{2}}(2\pi n l) \cos(\frac{n\pi b}{a}), \end{aligned} \tag{5.7}$$

where $K_\nu(z)$ is a modified Bessel function, $l = \frac{\sqrt{|D|}}{2a}$.

To investigate the distribution of the higher distance energies $E_{D,k}(N)$, we initiate the study of higher moments of the Epstein zeta functions, i.e.

$$Z_{Q,k}(s) := \sum_{n=1}^{\infty} \frac{r_Q(n)^k}{n^s}, \ k \geq 3. \tag{5.8}$$

Question: Do these higher moments satisfy any functional equation or have closed formulae as of (5.6) or (5.7)?

If they do, then we should be able to derive asymptotics for the average of $r_D^k(n)$ by Perron's formula as in Lemma 3.3. It has been shown that $Z_Q(s)$ attains the minimum only for equivalent forms of Q_{-3}, i.e., for the hexagonal lattice, whenever $s \geq 0$, see Cassels [1]. Thus, we wonder if this is true for all the higher moments and suggest the following

Conjecture 5.5. *For all $k \geq 1$, $Z_{Q,k}(s) \geq Z_{Q_{-3},k}(s), \forall s > 1$. After analytic continuation (if there is), this should be true for all $s \geq 0$.*

Appendix A. A factor-out phenomenon in summations

In most cases, a summation over the product of arithmetic functions would not have the simple "factor-out" property in the form that $\sum fg = g \sum f$. We notice this property for some functions g of slow growth and introduce a double method to deal with such summations, especially for homogeneous factors. We first notice such a factor-out phenomenon in summations with a log factor and another homogeneous input, for which we introduce a double-counting method as follows

Lemma A.1. *If the function $f(x, y) > 0$ is homogeneous, i.e. $f(kx, ky) = f(x, y), \forall k \in \mathbb{R}$, and integrable in x, then*

$$\sum_{n \le N} f(n, N) \log n \sim cN \log N,$$

where $c = \int_0^1 f(x, 1)dx$.

Here $f(x) \sim g(x)$ always means $\frac{f(x)}{g(x)} \to 1$ as x tends to infinity. Results of this form seem new in the author's view, but they might have been used by other authors. Though clear enough by itself, we prove it by double-counting as follows

Proof. To address the summation, we introduce a double-counting method to separate the log factor as follows. First, we partition the interval $[1, N]$ into $[\frac{m-1}{K}N, \frac{m}{K}N)$ for $m = 1, \dots, K$. On each subinterval, since f is projective and continuous, we can easily squeeze the partial sum as

$$\xi_m \sum_{n=\frac{(m-1)N}{K}}^{\frac{mN}{K}} \log n < \sum_{n=\frac{(m-1)N}{K}}^{\frac{mN}{K}} f(n, N) \log n < \eta_m \sum_{n=\frac{(m-1)N}{K}}^{\frac{mN}{K}} \log n, \tag{A.1}$$

where $\xi_m = \min_{\frac{m-1}{K} \le \frac{n}{N} < \frac{m}{K}} \{f(\frac{n}{N}, 1)\}$ and $\eta_m = \max_{\frac{m-1}{K} \le \frac{n}{N} < \frac{m}{K}} \{f(\frac{n}{N}, 1)\}$. Then we can asymptotically approximate the partial sum of $\log n$ by an integral as

$$\sum_{n=\frac{(m-1)N}{K}}^{\frac{mN}{K}} \log n \sim \frac{N}{K} \int_{m-1}^{m} (\log x + \log \frac{N}{K}) dx \sim \frac{N}{K} (\log \frac{N}{K} + \log m) \sim \frac{N \log N}{K},$$

if we set $\log K = o(\log N)$, i.e. $K = N^{o(1)}$. Thus by (A.1), the sum may be abbreviated to

$$\frac{1}{N \log N} \sum_{n=1}^{N-1} f(n, N) \log n \sim \frac{1}{K} \sum_{m<K} \theta_m \sim \int_0^1 f(x, 1)dx,$$

for some $\xi_m \le \theta_m \le \eta_m$, if $f(x, 1)$ is (Riemann) integrable. □

If $f(x, y)$ is not homogeneous, but is "weighted" as $f(kx, ky) = k^\alpha f(x, y)$ for some $\alpha \in \mathbb{R}$, then (A.1) is just scaled by N^α and the result becomes

Corollary A.2. *If $f(x,y)$ is homogeneous of degree $\alpha \in \mathbb{R}$, i.e. $f(kx,ky) = k^\alpha f(x,y)$, and integrable in x, then*

$$\sum_{n \le N} f(n,N) \log n \sim cN^{1+\alpha} \log N,$$

where $c = \int_0^1 f(x,1)dx$.

For the most obvious example, let $f(x,y) = x^{1+\alpha}/y, \alpha > -2$. Then it just tells us that $\sum_{n \le N} n^{1+\alpha} \log n \sim \frac{1}{2+\alpha} N^{2+\alpha} \log N$, which is seen from an obvious approximation by an integral.

Moreover, the double counting method allows us to handle summation with other factors than just the log factor, provided that the factor behaves as well as in the following

Corollary A.3. *Suppose that the function $f(x,y)$ is homogeneous of degree $\alpha \in \mathbb{R}$ and integrable in x, and that $g(x)$ has the property that $g(N) \to \infty$ and $g(Nx) = g(N) + o(g(N))$ for $0 < \delta(N) < x < 1$ and $\delta(N) \to 0$ as $N \to +\infty$. Then*

$$\sum_{n \le N} f(n,N) g(n) \sim cNg(N),$$

where $c = \int_0^1 f(x,1)dx$.

Proof. Following the proof of Lemma A.1, the summation of logarithms is substituted by that of $g(n)$. By the property of $g(x)$, we have for $K = 1/\delta(N)$,

$$\begin{aligned} \sum_{\frac{m-1}{K} N \le n \le \frac{m}{K} N} g(n) &\sim \int_{\frac{m-1}{K} N}^{\frac{m}{K} N} g(x)dx = \frac{N}{K} \int_{m-1}^{m} g(Nx/K)dx \\ &= \frac{N}{K}(g(N) + o(g(N))) \sim \frac{Ng(N)}{K}. \end{aligned}$$

Thus,

$$\lim_{K \to +\infty} \frac{1}{Ng(N)} \sum_{n \le N} f(n,N) g(n) = \lim_{K \to +\infty} \frac{1}{K} \sum_{m < K} \theta_m = w \int_0^1 f(x,1)dx.$$

□

Remark A.4: If $g(x) = (\log x)^\beta$ for some $\beta > 0$, then $g(Nm/K) = (\log N + \log(m/K))^\beta \sim (\log N)^\beta$ for $m \le K$ and $\log K = o(\log N)$, so that similar to Lemma A.1 we have

$$\sum_{n \le N} f(n,N)(\log n)^\beta \sim cN(\log N)^\beta.$$

Moreover, it might be generalized to arbitrary $g(x)$ of slow growth by appropriate double counting. Also, it would be interesting to derive the minor terms of the above summations.

Appendix B. Asymptotics of the second distance energy for square lattices

Now we apply the above results to an explicit counting problem in discrete geometry or number theory. Actually, we found the homogeneous phenomena while studying the following problem. Let $P = [\sqrt{N}] \times [\sqrt{N}]$ be the square grid of size N, where $[x]$ denotes the set of integers ranging from 1 to $\lfloor x \rfloor$. By studying the value distribution of $x^2 + y^2$ on P, it can be estimated that $d(P) := |\{d(p,q) \mid p,q \in P\}| \sim c\frac{|P|}{\sqrt{\log |P|}}$ for some $c > 0$. This becomes the initiating example for the Erdős conjecture on distinct distances in the Euclidean plane $\mathbb{R}^2$, which says $d(P) := |\{d(p,q) \mid p,q \in P\}| \geq c\frac{|P|}{\sqrt{\log |P|}}$ for any finite set $P \subset \mathbb{R}^2$ and some absolute constant $c > 0$.

Guth and Katz [4] established the nearly optimal bound $d(P) \geq c\frac{|P|}{\log |P|}$. The essential object therein is what they call *distance quadruples*, i.e.

$$Q(P) =: \{(p_1, q_1, p_2, p_2) \in P^4 \mid d(p_1, q_1) = d(p_2, q_2)\}.$$

We call $|Q(P)|$ the *distance energy* of P, denoted by $E_2(P)$. Note that in the appendix of [4], $E_2([\sqrt{N}]\times[\sqrt{N}])$ is estimated to be $\theta(N^3 \log N)$ by counting line-line incidences in $\mathbb{R}^3$. In this section, we establish the asymptotics of $E_2(P)$ for P being square lattices in circles, resorting to our homogeneous method.

Denote $r(n) := |\{(a,b) \in \mathbb{Z}^2 \mid a^2 + b^2 = n\}|$. On average, we have the following estimate:

Lemma B.1 (see (7.20) of Wilson [15]). *For any positive integer k and $x \in \mathbb{R}_+$, we have*

$$\sum_{n \leq x} r^2(n) \sim 4x \log x + O(x).$$

The precise constant of x in the above estimate can be found in Ramanujan [12]. A more precise estimate of the distance energy on square grids takes us more effort to develop number-theoretic methods. For convenience, we study lattice grids in circles, i.e., $P = \mathbb{Z}^2 \cap B_{\sqrt{N}}(0,0)$, where $B_n(a,b)$ denotes the disk centered at (a,b) with radius n. By results of the Gauss circle problem (see 1.4 of [8]),

$$|P| = \pi N + o(N^{1/3}). \tag{B.1}$$

Denote by $r_{a,b}(n) = \{(x,y) \in P \mid (x-a)^2 + (y-b)^2 = n\}$ so that $r_{0,0}(n) = r(n)$ for $n \leq N$. Actually, if $\sqrt{a^2+b^2} \leq \sqrt{N} - \sqrt{n}$, then $r_{a,b}(n) = r(n)$. For $\sqrt{a^2+b^2} > \sqrt{N} - \sqrt{n}$, $\partial B_{\sqrt{n}}(a,b)$ is cut by $\partial B_{\sqrt{N}}(0,0)$. By easy calculation, the cut arc has angle $2\arccos\left(\frac{a^2+b^2+n-N}{2\sqrt{n(a^2+b^2)}}\right)$. Then by symmetry, one may expect that

$$r_{a,b}(n) \sim \tilde{r}_{a,b}(n) := \begin{cases} r(n), \text{ if } \sqrt{a^2+b^2} \le \sqrt{N} - \sqrt{n}, n \le N; \\ 0, \text{ if } \sqrt{a^2+b^2} \le \sqrt{n} - \sqrt{N}, n > N; \\ \frac{r(n)}{\pi} \arccos\left(\frac{a^2+b^2+n-N}{2\sqrt{n(a^2+b^2)}}\right), \text{ otherwise.} \end{cases} \tag{B.2}$$

Although the estimate by $\tilde{r}_{a,b}(n)$ may deviate from the true distribution, the summation $R(n) := \sum_{(a,b)\in P} r_{a,b}(n)$ counting all the pairs of points $(p,q) \in P^2$ with $d(p,q) = n$, turns out to be valid from the average symmetric point of view. We may use area counting to clarify this. Define $s_{a,b}(n) = |\{(x,y) \in \mathbb{Z}^2 \mid (x-a)^2 + (y-b)^2 \le n\}$ for any $(a,b) \in B_{\sqrt{N}}(0,0), 0 \le n \le 4N$. Denote by $s(n) = s_{0,0}(n)$. Clearly by simple trigonometry

$$s_{a,b}(n) - s_{a,b}(n-1) = \frac{s(n) - s(n-1)}{\pi} \arccos\left(\frac{a^2+b^2+n-N}{2\sqrt{n(a^2+b^2)}}\right) + O(1).$$

Hence we have

$$\begin{aligned} R(n) = S(n) - S(n-1) &= \sum_{a^2+b^2\le N} (s_{a,b}(n) - s_{a,b}(n-1)) \\ &= \sum_{\sqrt{a^2+b^2}\le\sqrt{N}-\sqrt{n}} r(n) + \sum_{\sqrt{a^2+b^2}>\sqrt{N}-\sqrt{n}} \tilde{r}_{a,b}(n) + O(N). \end{aligned} \tag{B.3}$$

More explicitly, we show

Lemma B.2. *Let P be the integer points in the disk of radius $\sqrt{N}$ and $R(n)$ be the number of pairs of points from P with distance $\sqrt{n}, n \le 4N$ as above. Then*

$$R(n) = \left(2\arccos\left(\frac{\sqrt{n/N}}{2}\right) - \sqrt{\frac{4Nn - n^2}{4N^2}}\right) N r(n) + O(N).$$

Proof. By (B.1), (B.2) and (B.3), we have for $n \le N$,

$$\begin{aligned} R(n) = r(n) &\sum_{\sqrt{a^2+b^2}\le\sqrt{N}-\sqrt{n}} 1 \\ &+ \frac{r(n)}{\pi} \sum_{\sqrt{N}-\sqrt{n}<\sqrt{a^2+b^2}\le\sqrt{N}} \arccos\left(\frac{a^2+b^2+n-N}{2\sqrt{n(a^2+b^2)}}\right) + O(N) \\ &= \pi r(n)(\sqrt{N}-\sqrt{n})^2 \\ &+ \frac{r(n)}{\pi} \iint_{(\sqrt{N}-\sqrt{n})^2\le x^2+y^2\le N} \arccos\left(\frac{x^2+y^2+n-N}{2\sqrt{n(x^2+y^2)}}\right) dxdy + O(N). \end{aligned}$$

Using the polar coordinates we may transform the double integral into

$$2\pi \int_{\sqrt{N}-\sqrt{n}}^{\sqrt{N}} r \arccos\left(\frac{r^2+n-N}{2\sqrt{n}r}\right) dr$$

$$=\pi r^2 \arccos\left(\frac{r^2+n-N}{2\sqrt{n}r}\right)\Big|_{\sqrt{N}-\sqrt{n}}^{\sqrt{N}} + \pi\int_{\sqrt{N}-\sqrt{n}}^{\sqrt{N}} r^2 \frac{\frac{1}{2\sqrt{n}}+\frac{N-n}{2\sqrt{n}r^2}}{\sqrt{1-\frac{(r^2+n-N)^2}{4nr^2}}}dr$$

$$=\pi N \arccos\left(\frac{\sqrt{n/N}}{2}\right) - \pi^2(\sqrt{N}-\sqrt{n})^2$$

$$+\frac{\pi}{2}\int_{\sqrt{N}-\sqrt{n}}^{\sqrt{N}} \frac{r^2+N-n}{\sqrt{4nr^2-(r^2+n-N)^2}}d(r^2).$$

Substituting by $s=\frac{r^2-n-N}{2\sqrt{Nn}}$ we get

$$\int_{-1}^{-\frac{\sqrt{n/N}}{2}} \frac{2\sqrt{Nn}s+2N}{\sqrt{1-s^2}}ds$$

$$=-2\sqrt{Nn}\sqrt{1-s^2}\,\Big|_{-1}^{-\frac{\sqrt{n/N}}{2}} + 2N\arcsin(s)\,\Big|_{-1}^{-\frac{\sqrt{n/N}}{2}}$$

$$=-\sqrt{4Nn-n^2}+2N\left(\frac{\pi}{2}-\arcsin\left(\frac{\sqrt{n/N}}{2}\right)\right).$$

Summing up everything provides us for $n\le N$,

$$R(n)=\pi r(n)(\sqrt{N}-\sqrt{n})^2 + r(n)N\arccos\sqrt{\frac{n}{2N}} - \pi r(n)(\sqrt{N}-\sqrt{n})^2$$

$$-\frac{r(n)}{2}\sqrt{4Nn-n^2}+\frac{\pi r(n)}{2}N - r(n)N\arcsin\sqrt{\frac{n}{2N}}+O(N)$$

$$=r(n)\left(N\arccos\sqrt{\frac{n}{2N}}-\sqrt{Nn-\frac{n^2}{4}}+\frac{\pi}{2}N-N\arcsin\sqrt{\frac{n}{2N}}\right)$$

$$+O(N)$$

$$=\left(2\arccos\left(\frac{\sqrt{n/N}}{2}\right)-\sqrt{\frac{4Nn-n^2}{4N^2}}\right)Nr(n)+O(N).$$

When $N<n\le 4N$, we have by (B.2)

$$R(n)=\frac{r(n)}{\pi}\sum_{\sqrt{n}-\sqrt{N}<\sqrt{a^2+b^2}\le\sqrt{N}} \arccos\left(\frac{a^2+b^2+n-N}{2\sqrt{n(a^2+b^2)}}\right)+O(N)$$

$$=\frac{r(n)}{\pi}\iint_{(\sqrt{N}-\sqrt{n})^2\le x^2+y^2\le N} \arccos\left(\frac{x^2+y^2+n-N}{2\sqrt{n(x^2+y^2)}}\right)dxdy+O(N)$$

$$=r(n)r^2\arccos\left(\frac{r^2+n-N}{2\sqrt{n}r}\right)\Big|_{\sqrt{n}-\sqrt{N}}^{\sqrt{N}}$$

$$+\frac{r(n)}{2}\int_{\sqrt{n}-\sqrt{N}}^{\sqrt{N}} r^2\frac{\frac{1}{2\sqrt{n}}+\frac{N-n}{2\sqrt{n}r^2}}{\sqrt{1-\frac{(r^2+n-N)^2}{4nr^2}}}dr+O(N)$$

$$
\begin{aligned}
&= Nr(n) \arccos\left(\frac{\sqrt{n/N}}{2}\right) \\
&\quad + \frac{r(n)}{2} \int_{\sqrt{n}-\sqrt{N}}^{\sqrt{N}} \frac{r^2+N-n}{\sqrt{4nr^2-(r^2+n-N)^2}} d(r^2) + O(N) \\
&= Nr(n) \arccos\sqrt{\frac{n}{4N}} \\
&\quad + \frac{r(n)}{2}\left(-\sqrt{4Nn-n^2} + 2N\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{n}{4N}}\right)\right) + O(N) \\
&= \left(2\arccos\sqrt{\frac{n}{4N}} - \sqrt{\frac{4Nn-n^2}{4N^2}}\right) Nr(n) + O(N),
\end{aligned}
$$

which adopts the same form as for $n \leq N$. □

As asymptotics of a single $R(n)$, the above result seems too weak, but it provides us with the main term of the distance energy as follows

Theorem B.3. *Let P be the set of integral lattice points in a disk of radius $\sqrt{N}$, then*

$$E_2(P) \sim (4\pi^2 - 8\pi + 16) N^3 \log N.$$

Proof. Let $E(x) = \sum_{n \leq x} r^2(n)$. Then by Lemma B.1, Lemma B.2, (3.2) and Abel's summation by parts, we get (noting that $\sum_{n \leq N} r(n) \sim \pi N$)

$$
\begin{aligned}
E_2(P) &= \sum_{n \leq 4N} R(n)^2 \\
&= N^2 \sum_{n \leq 4N} r^2(n) \left(2\arccos\sqrt{\frac{n}{4N}} - \sqrt{\frac{4Nn-n^2}{4N^2}}\right)^2 + O(N^3) \\
&= 4N^2 \sum_{n=1}^{4N} r^2(n) \arccos^2\sqrt{\frac{n}{4N}} \\
&\quad - 2N^2 \sum_{n=1}^{4N} r^2(n) \sqrt{\frac{4Nn-n^2}{N^2}} \arccos\sqrt{\frac{n}{4N}} \\
&\quad + \frac{N^2}{4} \sum_{n=1}^{4N} r^2(n) \frac{4Nn-n^2}{N^2} + O(N^3) \\
&= I - II + III + O(N^3).
\end{aligned}
$$

Using Abel summation and Lemma B.1, we get

$$I = 4N^2 \sum_{n=1}^{4N} (E(n) - E(n-1)) \arccos^2\sqrt{\frac{n}{4N}}$$

$$
\begin{aligned}
&=4N^2\sum_{n=1}^{4N-1}E(n)\left(\arccos^2\sqrt{\frac{n}{4N}}-\arccos^2\sqrt{\frac{n+1}{4N}}\right)+O(N^2)\\
&=8N^2\sum_{n=1}^{4N-1}E(n)\arccos\sqrt{\frac{n}{4N}}\left(\arccos\sqrt{\frac{n}{4N}}-\arccos\sqrt{\frac{n+1}{4N}}\right)+O(N^2)\\
&=4N^2\sum_{n=1}^{4N-1}\frac{E(n)}{\sqrt{(4N-n)n}}\arccos\sqrt{\frac{n}{4N}}+O(N^2\log N)\\
&=16N^2\sum_{n=1}^{4N-1}\frac{\sqrt{n}\log n}{\sqrt{4N-n}}\arccos\sqrt{\frac{n}{4N}}+O(N^3).
\end{aligned}
$$

Now the above summation falls into the case of Lemma A.1, which shows

$$I \sim 16c_1N^3\log N,\ c_1=\int_0^1\sqrt{\frac{t}{1-t}}\arccos(\sqrt{t})dt=\frac{\pi^2-4}{8}.$$

Similarly by Abel summation,

$$
\begin{aligned}
II=&2N^2\sum_{n=1}^{4N}(E(n)-E(n-1))\sqrt{\frac{4Nn-n^2}{N^2}}\arccos\sqrt{\frac{n}{4N}}\\
=&2N^2\sum_{n=1}^{4N-1}E(n)\left(\sqrt{\frac{4Nn-n^2}{N^2}}\arccos\sqrt{\frac{n}{4N}}\right)-\\
&2N^2\sum_{n=1}^{4N-1}E(n)\left(-\sqrt{\frac{4N(n+1)-(n+1)^2}{N^2}}\arccos\sqrt{\frac{n+1}{4N}}\right)+O(N^2)\\
=&8N^2\sum_{n=1}^{4N-1}\left(\frac{1-2\sqrt{\frac{n}{4N}}}{\sqrt{1-\frac{n}{4N}}}\arccos\sqrt{\frac{n}{4N}}-\sqrt{\frac{n}{4N}}\right)\log n+O(N^3)\\
\sim&8(c_2-\frac{2}{3})N^3\log N,
\end{aligned}
$$

where $c_2=\int_0^1\frac{1-2\sqrt{t}}{\sqrt{1-t}}\arccos(\sqrt{t})dt=\pi-2-2c_1$. And also,

$$
\begin{aligned}
III=&\frac{N^2}{4}\sum_{n=1}^{4N}(E(n)-E(n-1))\frac{4Nn-n^2}{N^2}\\
=&\frac{N^2}{4}\sum_{n=1}^{4N-1}E(n)\left(\frac{4Nn-n^2}{N^2}-\frac{4N(n+1)-(n+1)^2}{N^2}\right)+O(N^2)\\
=&\sum_{n=1}^{4N-1}(2n-4N+1)n\log n+O(N^3)\\
\sim&\frac{32}{3}N^3\log N.
\end{aligned}
$$

Finally, altogether we get

$$\begin{aligned} E_2(P) &\sim I - II + III \sim (16c_1 - 8c_2 + 16)N^3 \log N \\ &= (4\pi^2 - 8\pi + 16)N^3 \log N. \end{aligned}$$

□

Remark B.4: Notice that the above summations (divided by $N \log N$) converge extremely slowly. For say the last summation in I, computing until $N = 10^{11}$, the second decimal is not even stable.

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Shenzhen MSU-BIT University & Guangdong Laboratory of Machine Perception and Intelligent Computing, Shenzhen, Guangdong 518172, China
E-mail address: `zhipeng.lu@hotmail.com`

School of Mathematics, Shandong University, Jinan, Shandong, China
E-mail address: `xianchang.meng@gmail.com`