# COLOURINGS OF APERIODIC TILINGS

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ABSTRACT. We find explicit optimal vertex, edge and face coulourings for the chair tiling, the Ammann–Beenker tiling, the rational pinwheel tiling and the pinwheel tiling.

## 1. INTRODUCTION

Many nice models of point sets with long-range aperiodic order are produced via substitution rules. A tile substitution in the plane, with polygonal prototiles, produces a tiling of $\mathbb{R}^2$, which can be seen as a planar drawing of an infinite planar graph (a plane graph) whose faces are the tiles. See [1] for the theory of substitution tilings and the Tilings Encyclopedia [4] for a wonderful graphical catalogue of known substitution tilings. It is natural to ask what is the chromatic number, the chromatic index of this infinite graph, and what is the optimal face colouring number for this graph (or equivalently, the chromatic number of the dual graph). A question that, to date, has only been answered in the rhombic Penrose tiling [10] to the best of the authors' knowledge. We answer these questions for some well-known 2-dimensional substitution tilings with explicit colouring algorithms.

As all our tilings are infinite planar graphs, the 4-colour theorem gives us an upper-bound of four for both the chromatic number and the optimal number of colours used for face colouring. It is easy to check when the lower bound of two is reached (or more precisely, not reached), typically leaving us with only two potential choices. Similarly, Vizing's theorem (Corollary 2.5) gives us only two potential choices for the chromatic index. While this seems to make the problems easy, deciding between the two choices is in general a hard problem. For example, deciding between the two choices for the

chromatic index for finite graphs is an NP-complete problem [7], and we are dealing with infinite graphs.

The paper is organized as follows: Section 2 contains some preliminary material on finite and infinite planar graph colouring. Section 3 proves that the chair tiling has chromatic number 2, chromatic index 4, and a minimal face colouring of 3. Section 4 proves that the Ammann–Beenker tiling has a chromatic number 3, chromatic index 8, and a minimal face colouring of 2. Section 5 proves that the rational pinwheel tiling has chromatic number 3, chromatic index 8, and a minimal face colouring of 3. Finally, Section 6 proves that the pinwheel tiling has chromatic number 3, chromatic index 8, and a minimal face colouring of 3. This can all be summarised in the following table.

| Tiling | $\chi_0$ | $\chi_1$ | $\chi_2$ |
|---|---|---|---|
| chair | 2 | 4 | 3 |
| Ammann–Beenker | 3 | 8 | 2 |
| rational pinwheel | 3 | 8 | 3 |
| pinwheel | 3 | 8 | 3 |

As mentioned before, all of these colourings are by way of explicit algorithms.

## 2. Preliminaries

In this paper we will look at the planar graph induced by tilings of the plane with polygonal tiles where tiles meet full edge to full edge. While other tilings of the plane are possible, this is a good starting point. We start by reviewing in this section some of the basic properties of colourings of (infinite) graphs.

Recall that a graph is called planar if it can be drawn in the plane in such a way that the edges only meet at vertices. In this case, any such particular drawing is called a plane graph. An edge $e$ is called a *bridge* if it belongs to no cycle. All the tilings we study in this paper can be seen as infinite plane graphs, with all faces finite and without bridges. In particular, the dual graph of any tiling will be an infinite simple plane graph.

Let us start by recalling the following definitions of [12, Chapter 5].

**Definition 2.1.** *Let $G$ be a (not necesarily finite) simple graph*[1].

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[1]Many books, including [12], denote by $\chi(G)$ and $\chi'(G)$ the chromatic number and chromatic index, respectively. We use here the standard alternate notation $\chi_0$ for the vertices (0-dimensional objects of the graph) and $\chi_1$ for the edges (1-dimensional objects of the graph). This notation will be very helpful when we talk about the minimal number of colours needed to colour the faces of a plane graph.

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(a) *The* chromatic number $\chi_0(G)$ *is the smallest number of colours needed to colour the vertices of $G$ such that whenever $uv$ is an edge in $G$, the vertices $u, v$ have different colours.*

(b) *The* chromatic index $\chi_1(G)$ *is the smallest number of colours needed to colour the edges of $G$ such that whenever $e_1, e_2$ are edges which meet at a vertex, they have different colours.*

(c) *If $G$ is a plane graph whose dual $G^*$ is simple, we denote by $\chi_2(G)$ the smallest number of colours needed to colour the faces of $G$ such that, if two faces share an edge, they have different colours.*

Note here that $\chi_2(G) = \chi_0(G^*)$.

We denote by $\Delta(G)$ (or simply $\Delta$) the supremum of its degrees, that is

$$\Delta(G) := \sup\{\deg(v) : v \in V(G)\} \, .$$

We are particularly interested in tilings with finitely many tiles/prototiles. Note that for any such tiling $\Delta(G)$ must be finite.

We start by recalling one of the fundamental results about colouring infinite graphs, the de Bruijn–Erdős Theorem.

**Theorem 2.2** (de Bruijn–Erdős Theorem [6])**.** *Let $G$ be an infinite graph. Then*

(a) *The chromatic number $\chi_0(G)$ of $G$ is the supremum of the chromatic numbers of its finite subgraphs.*

(b) *The chromatic index $\chi_1(G)$ of $G$ is the supremum of the chromatic indices of its finite subgraphs.*

The de Bruijn–Erdős Theorem has many useful applications, some of which we list below.

Since a finite graph is 2-colourable if and only if it has no odd cycle [12, Thm. 2.1], we immediately get:

**Corollary 2.3.** *Let $G$ be an infinite plane graph, having only finite faces. Then $G$ is bipartite if and only if each face has an even number of edges. In particular, a tiling of the plane defines a bipartite graph if and only if all tiles have an even number of edges.*

*Proof.* It follows from Thm. 2.2 that $G$ is bipartite if and only if all finite cycles are even. We claim that all finite cycles are even if and only if each face has an even number of edges. The direct implication is obvious, while the converse follows immediately by dividing any finite cycle into faces, and observing that the total number of edges of all faces is the number of edges of the finite cycle plus twice the common edges to the faces. $\square$

By combining Thm. 2.2 with the famous 4-colour theorem (see for example [12, Thm. 5.5 and Cor. 5.10]) we get:

**Corollary 2.4.** *Let $G$ be an infinite plane graph without bridges. Then,*

    (a) *The vertices of $G$ can be 4-coloured.*
    (b) *The faces of $G$ can be 4-coloured.*

*Proof.* (a) This follows from Thm. 2.2 and the 4-colour theorem.

(b) Since $G$ is a plane graph without bridges, its dual $G^*$ is also a plane graph without loops. It follows from (a) that the vertices of $G^*$ can be 4-coloured. $\qquad\square$

Next, the Vizing theorem for finite graphs [12, Thm. 5.15] yields:

**Corollary 2.5** (Vizing theorem for infinite graphs)**.** *Let $G$ be an infinite graph. Then,*

    (a) *If $\Delta = \infty$ then $\chi_1(G) = \infty$.*
    (b) *If $\Delta < \infty$ then $\Delta \leq \chi_1(G) \leq \Delta + 1$.*

Exactly as for finite graphs [12, Thm. 5.18], the chromatic index of a bipartite graph $G$ is equal to $\Delta(G)$:

**Theorem 2.6** (König Theorem for infinite graphs)**.** *Let $G$ be an infinite bipartite graph. Then $\chi_1(G) = \Delta$.*

Finally, we show that for graphs with large $\Delta$, the chromatic index coincides with $\Delta$.

**Theorem 2.7.** *Let $G$ be an infinite planar graph. If $\Delta \geq 7$ then*

$$\chi_1(G) = \Delta \ .$$

*Proof.* The case $\Delta \geq 8$ is proven in [11]. The case $\Delta = 7$ follows from [9]. $\qquad\square$

2.1. **A simple tool for edge colourings of tilings.** Many of the edge colourings we will provide below are done via a simple method that we describe in full generality here.

**Theorem 2.8** (Directional alternating colouring method)**.** *Let $G$ be a simple plane graph which is finite or infinite, with all edges drawn as straight-line segments.*

*If there exists pairwise nonparallel vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$ such that each edge $e \in E(G)$ is parallel to some $\vec{u_j}$ then,*

$$\chi_1(G) \leq 2n \ .$$

*Moreover, an explicit edge colouring of $G$ with colours $c_1, c_2, \ldots, c_{2n}$ is given by the following method, which we will call the* directional alternating colouring method*:*

*Let $L$ be the set of all lines in $\mathbb{R}^2$ which contain at least an edge of $G$. For each $l \in L$, we know that there exists a unique $1 \leq j \leq n$ such that $l \parallel \vec{u_j}$. Colour all edges on $l$ alternately, no matter if they are adjacent or not, with colours $c_{2j-1}$ and $c_{2j}$.*

*Proof.* We need to show that the directional alternating colouring method produces a proper edge colouring.

Let $e_1, e_2$ be two edges that are adjacent at vertex $v$. Let $l_1, l_2$ be the lines containing $e_1, e_2$ and let $\vec{u_j}, \vec{u_k}$ be the unique vectors parallel to $l_1$ and $l_2$, respectively.

If $j \neq k$ then $e_1, e_2$ are coloured by a different colour.

If $j = k$ then $l_1 \parallel l_2$. Since both lines pass through $v$ we have $l_1 = l_2$. Therefore, $e_1, e_2$ are consecutive edges on $l_1$, and hence are coloured with different colours.

$\square$

## 3. The chair tiling

In this section we study the chair tiling. For background, we refer the reader to [1, 2, 5, 4].

The chair tiling refers to the fixed-point tiling of the plane produced by the following single tile substitution.
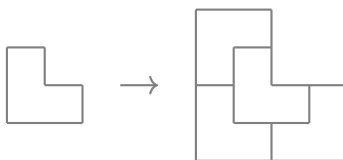


There are two natural choices for vertices. One choice is to let every tile have 8 vertices



while the other choice is to only insert vertices where 90-degree edges meet, or equivalently, only inserting vertices of degree at least 3, making the graph a map. We use the first choice, which makes the chair tiling bipartite and thus 2-colourable.

**Proposition 3.1.** *The chair tiling has chromatic index $\chi_1(C) = 4$.*

*Proof.* One can see in the picture following this proposition that $\Delta = 4$ for the chair tiling. Thus, an explicit four colouring of the edges is given by the directional alternating colouring method (Thm. 2.8) for the vectors $\vec{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\square$

*Remark 3.2:* The 8-vertex tile choice is bipartite by Cor. 2.3 and hence $\chi_1(G) = \Delta(G) = 4$ by Thm. 2.6. This provides an alternate nonconstructive proof of Prop. 3.1, for the 8-vertex version.

It is important to note that our proof of Prop. 3.1 works for both choices of vertices we mentioned for the chair tiling.

An edge colouring of a large patch of the 8-vertex tile choice chair tiling can be seen in Figure 1.
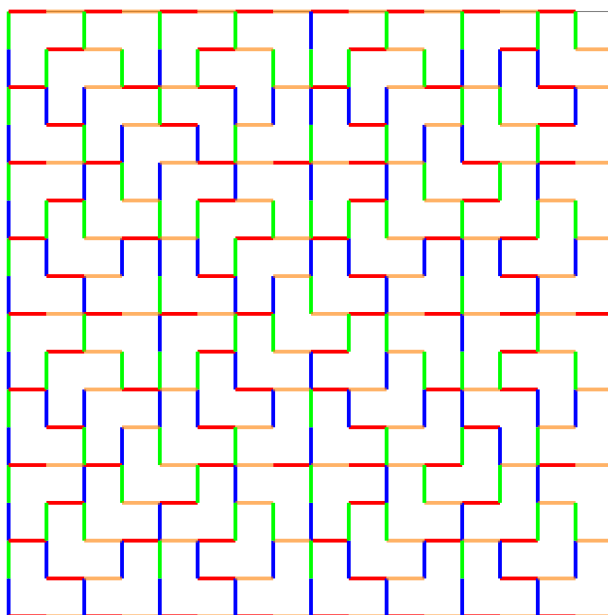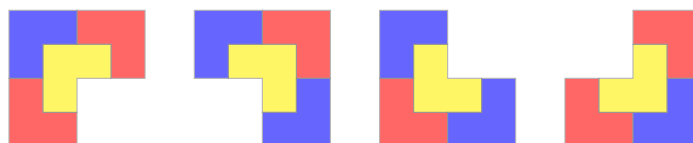


FIGURE 1. Edge colouring of a large patch of the 8-vertex tile choice chair tiling

**Theorem 3.3.** *Let C denote the chair tiling. Then, $\chi_2(C) = 3$. Moreover, a 3-colouring of the faces can be obtained by colouring the four first-level supertiles, based on orientation, the following way:*



*Proof.* Consider the tiling of the plane with level 2 supertiles. Using the hypothesized colourings, the colouring of the four second-level supertiles is included in Figure 2.
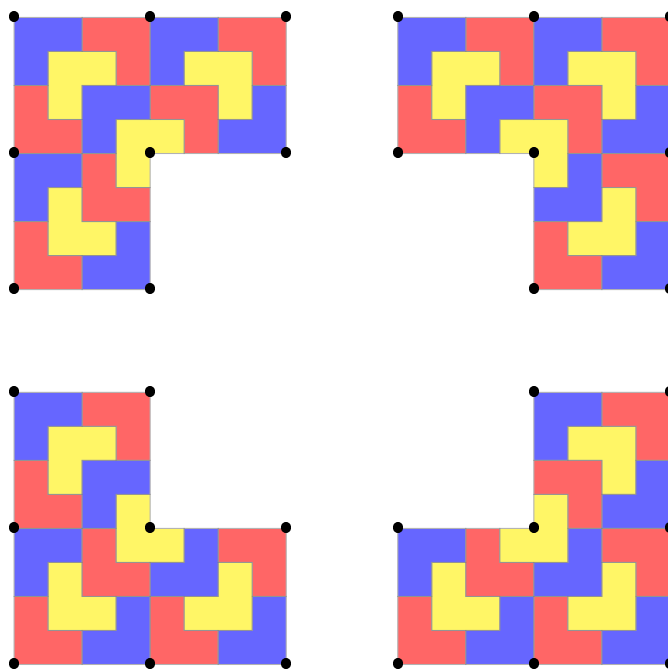
FIGURE 2. Colourings of the level 2 supertiles of the chair tiling.

We will call the segment between two markings a super-edge. Each super-edge is made up of two or three edges.

It is clear that level two supertiles must meet super-edge to super-edge, and that each super-edge is either horizontal or vertical. We show now that there is no conflict.

For each horizontal super-edge, by inspecting the four colourings of the super-tiles we see that the colour below along the super-edge is one of the following



Similarly, for each horizontal super-edge, the colours above along the super-edge are one of the following



By inspection, the only possible conflicts are the yellow colours for the following two possible configurations:

We claim that these two configurations are not possible (due to the 270°
angle appearing at the yellow colour), and hence no horizontal conflict oc-
curs.

Let us first look at the first possible conflict.

These colours occur above a super-edge only in one level 2 super-tile, and
the yellow tile above the super-tile is in the following position:

In the same way, the yellow tile above the super-tile is in the following
position:

Then, the two yellow tiles will intersect without completely overlapping,
which is not possible.

Similarly, in the second potential conflict, the two yellow tiles would be
in the following positions:

Again, this situation is not possible, as it would lead to two tiles partially
overlapping. We therefore have no horizontal conflict.

Vertical conflicts are eliminated via a similar analysis. Or alternately, we
can eliminate the vertical conflicts by observing that rotating each coloured
level 2 super-tile by 90° or 270° gives one of our coloured level 2 super-tiles
with the colours red and blue interchanged. It follows from here that rotating
our entire tiling by 90° simply interchanges the red and blue colours in our
colouring scheme. Then, by the first part of the argument, after rotating the
tiling by 90° we have no horizontal conflict, and hence the original colouring
scheme has no vertical conflict.                                    □

A larger patch of this face colouring, which highlights its aperiodic struc-
ture, is included in Figure 2.

*Remark 3.4:* As pointed out by the anonymous reviewer, there is a beautiful
diagonal structure in the 3 colouring from Figure 3.

In the direction of the first bisector, the diagonals alternate between a
diagonal of blue tiles meeting at corners and a diagonal of alternating yellow
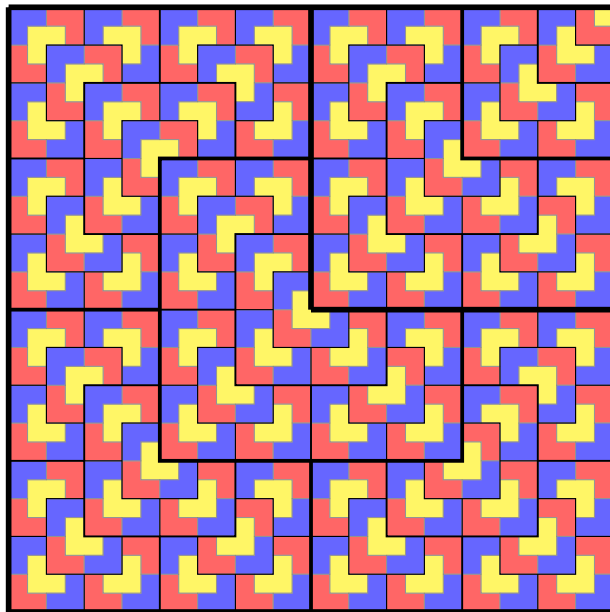and red tiles.

FIGURE 3. Large patch of colored chair tiling

In the perpendicular direction, the diagonals alternate between a diagonal of red tiles meeting at corners and a diagonal of alternating yellow and blue tiles.

The origin of this structure can be seen in Figure 2.

## 4. AMMANN–BEENKER TILING

The Ammann–Beenker tiling is a substitution tiling with two prototiles, an isosceles right triangle with side lengths $1, 1, \sqrt{2}$ and a rhombus of side 1 and with an angle of 45°, and the following inflation rule:



For more details about this tiling, we recommend [1, 3, 4].

We start by finding the chromatic number of the Ammann–Beenker tiling.

**Theorem 4.1.** *Let AB denote the Ammann–Beenker tiling. Then $\chi_0(AB) = 3$. An explicit 3-colouring can be induced by a two colouring of the borders of the second-level supertiles.*

*Proof.* Consider the sub-graph of AB generated by the borders of second-level super tiles. The borders of the rhombuses are 20-cycles and thus bipartite. For the triangles, while their borders are not bipartite, they always

come in pairs forming squares (because the $\sqrt{2}$ diagonals need to match up) that have a bipartite border. Thus, we can 2-colour these borders and 3-colour the interiors:
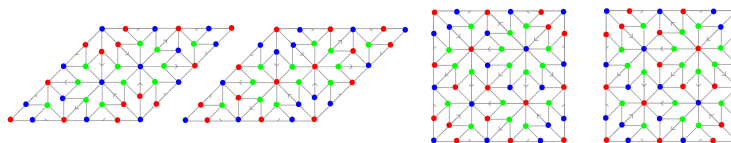


Since there are no conflicts inside, it follows that the AB tiling is 3-colourable.



$\square$
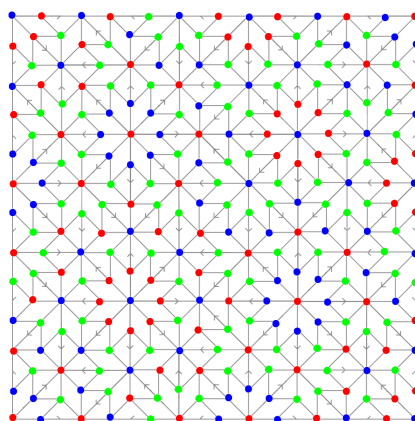
Next, we look at the chromatic index.

**Proposition 4.2.** *Let AB denote the Ammann–Beenker tiling, then $\chi_1(AB) = 8$. An explicit four colouring of the edges of AB is given by the directional alternating colouring method (Thm. 2.8) for the vectors $\vec{u}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $\vec{u}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\vec{u}_3 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $\vec{u}_4 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.*

*Proof.* Since the orientation of every edge is an integer multiple of $45°$, there are four orientations for the edges. The AB tiling has a vertex of degree eight; thus, the colouring follows from Thm. 2.8. A colouring of all edges in a large patch is included in Figure 4. $\square$

We complete the section by colouring the faces of the tiling.

**Theorem 4.3.** *Let AB denote the Ammann–Beenker tiling. Then $\chi_2(AB) = 2$. Moreover, a 2-colouring of the faces can be obtained by colouring the prototiles based on their orientation.*
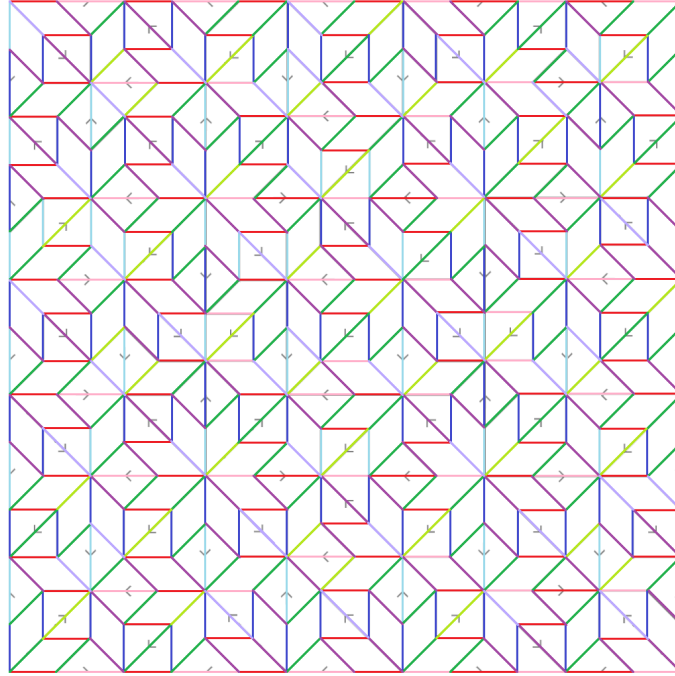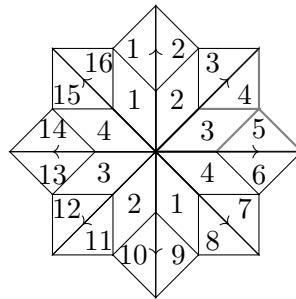
FIGURE 4. Edge-colourings of a large patch in the Ammann–
Beenker tiling

*Proof.* The rhombuses have four orientations, and taking into account the
directions of the arrows, the triangles have 16 orientations.



For the rhombuses, colour all those whose orientation is given by 1 and 3
blue. Colour 2 and 4 red. For triangles, colour the triangles whose orienta-
tion is given by 1, 4, 5, 8, 9, 12 and 16 red. Colour 2, 3, 6, 7, 10, 11, 14 and
15 blue. The end result is as follows:
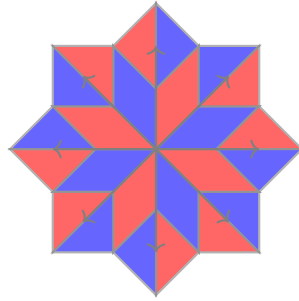
We shall show that this colouring scheme works. Focusing on just the interactions between triangles, all triangles whose common edge is the hypotenuse will be properly coloured due to the requirement that their arrows go in the same direction. Similarly, all triangles whose relative orientations are as shown below will also be properly coloured:



A potential issue with our colouring could occur if adjacent triangles had arrows pointing in opposite directions, but it is easy to show by our inflation rule that this does not happen:
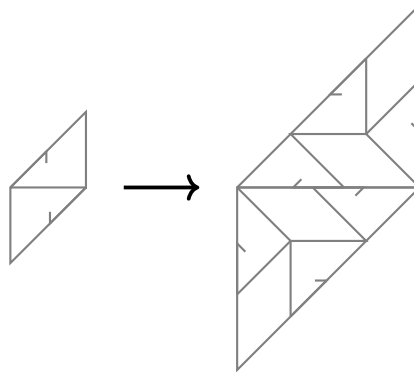


The hypotenuse of a triangle is never adjacent to an edge of the rhombus; thus the above triangle configuration does not occur.

Focusing on the interactions between rhombuses, by our colouring scheme, rhombuses which are the same colour differ by an angle of 90° or are the same orientation. Since the rhombuses whose orientation differs by 90° cannot be adjacent, it follows that the only possible conflict is between two rhombuses of the same orientation. It is easy to show by the inflation rule that this cannot happen:
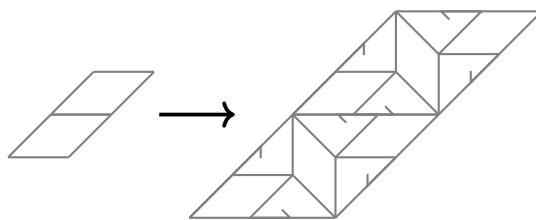
Finally, focusing on the interactions between rhombuses and triangles:
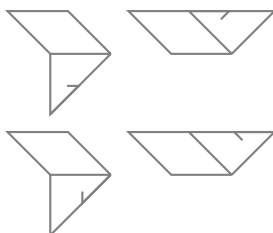
All possible combinations of adjacent triangles and rhombuses are just a rotated/reflected version of one of the above. It is easy to see that the first two configurations and their rotated/reflected versions work based on our colouring rule. The last two do not, but we can show by the inflation rule that the last two relative positions of triangles and rhombuses do not occur:
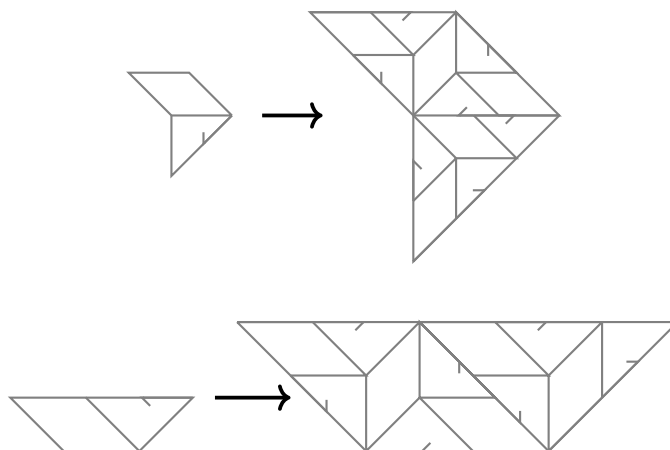
Since our cases are exhaustive, it follows that the faces of the AB tiling are two colourable. A 2-colouring of the faces in a large patch is included in Figure 5.                                                                                        □

A face two colouring can be shown by the degrees being even but the above argument produces a colouring algorithm. In fact this is another way to show that every vertex index is even (meaning all cycles are even in the dual graph).
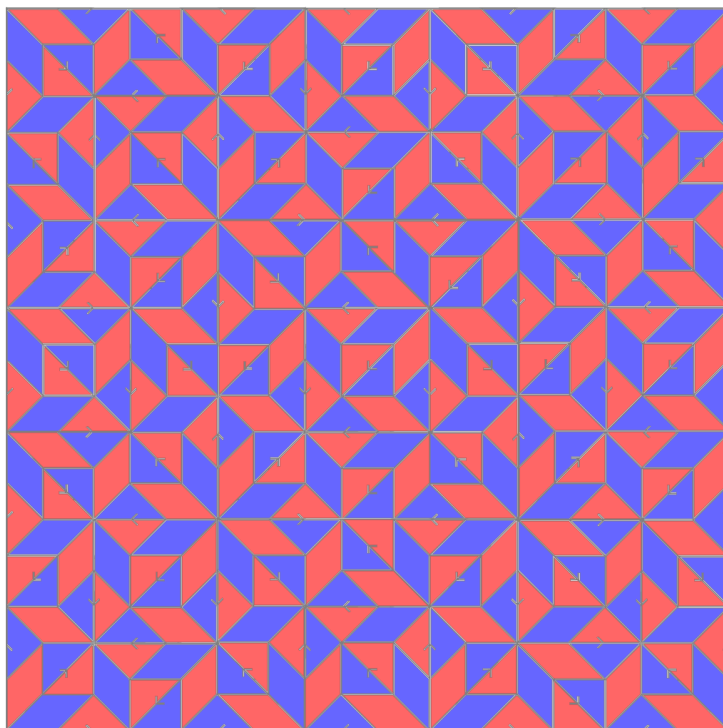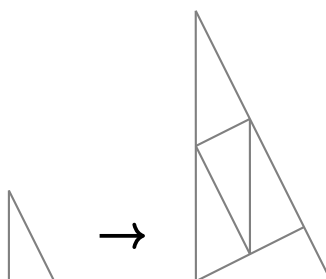
FIGURE 5. Face-colourings of a large patch in the Ammann–Beenker tiling
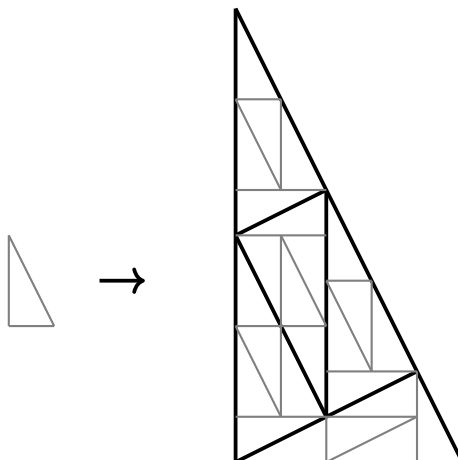
## 5. Rational pinwheel tiling

The rational pinwheel substitution is the single tile substitution:



Let RP denote the rational pinwheel tiling of the plane. The structure of RP is much clearer by observing that it is also created by the square of the

substitution:



Let us mention here in passing that the colourings we get in this section and the next section are obtained by looking locally at how a tile sits inside a (level 1 or level 2) supertile.

We start by proving the following preliminary Lemma.

**Lemma 5.1.** *The tiles of RP combine into rectangles, and all rectangles will appear in one of the following* two *orientations:*



*Proof.* All the tiles in this second substitution are the original triangle rotated by one of the following four angles: $0°, 90°, 180°, 270°$. As the sides have length $1, 2, \sqrt{5}$ and as $\sqrt{5}$ is irrational, the hypotenuse of any tile will coincide with only the hypotenuse of another tile.

Therefore, a simple induction shows that for all $n$, all tiles in the $2n$-supertiles always appear in one of these four rotations and must be paired with a corresponding $180°$ rotated triangle along its hypotenuse, unless it appears on the hypotenuse of the $2n$-supertile.                □

We can now find the chromatic number of the rational pinwheel tiling.

**Theorem 5.2.** *The chromatic number of RP is $\chi_0(RP) = 3$.*
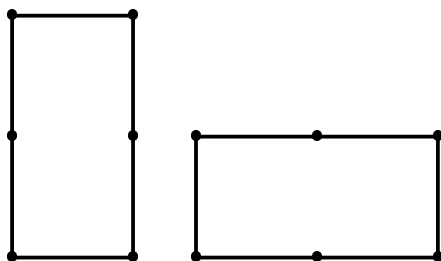
*Proof.* An explicit 3-colouring of the rational pinwheel can be obtained in the following way. From the previous lemma we know that the rational pinwheel is tiled by two rectangles that fit together in a very conventional way. If we replace

with their first-level supertiles
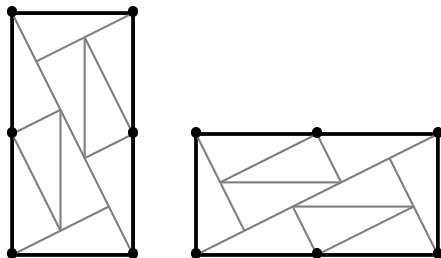


we see that the rational pinwheel has a subtiling by the rectangles



with these rectangles meeting at the nodes indicated. This forms a bipartite graph which can be two coloured. Hence, we can two colour these nodes and fill in the interiors using the following four tiles as in Figure 6.
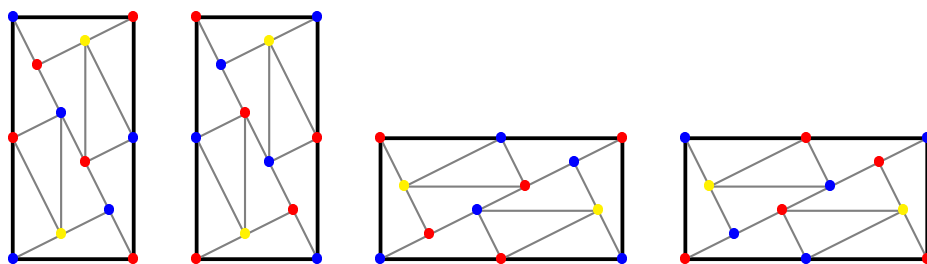


FIGURE 6. Vertex colouring of the supertiles in rational pinwheel

Since the rational pinwheel contains 3-cycles, it is not bipartite and hence $\chi(RP) = 3$. $\qquad\square$

Next we find the chromatic index.

**Proposition 5.3.** *The chromatic index of the rational pinwheel is* $\chi_1(RP) = 8$.

*Proof.* We first observe that RP has a vertex of degree 8 by looking at the central vertex of two rectangles formed by second-level supertiles:
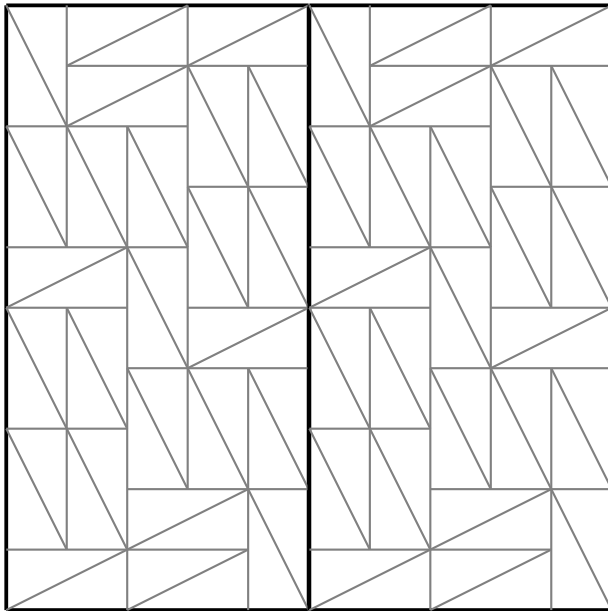


By Lemma 5.1 all the edges in the rational pinwheel are parallel to one of the vectors $(1,0), (0,1), (2,1)$ and $(-1,2)$. Therefore, an explicit 8 colouring of RP is given by the directional alternating colouring method with vectors $(1,0), (0,1), (2,1)$ and $(-1,2)$.                    $\square$

*Remark 5.4:* A nonconstructive proof of Prop. 5.3 is given by Thm. 2.7.

We complete the section by colouring the faces of the RP.

**Theorem 5.5.** *We have $\chi_2(RP) = 3$.*

*Proof.* An explicit 3-colouring of the faces of RP is given by colouring the second-level supertile versions of the two rectangles of Lemma 5.1 and showing that these tiles fit together with no conflicts. The colourings in question are given in Figure 7.

Each edge of these rectangles is made up of one or two hypotenuses of the level one supertile. We will split the hypotenuses on the boundaries of the tiles into two types:

- Type I: if the hypotenuse is on the bottom or left edge of the rectangle.
- Type II: if the hypotenuse is on the top or right edge of the rectangle.

Note that going counterclockwise around the boundary, each Type I hypotenuse is made up in order of exactly a short edge of a yellow triangle,

Type II

Type I

Type II

Type I

Type II

Type I
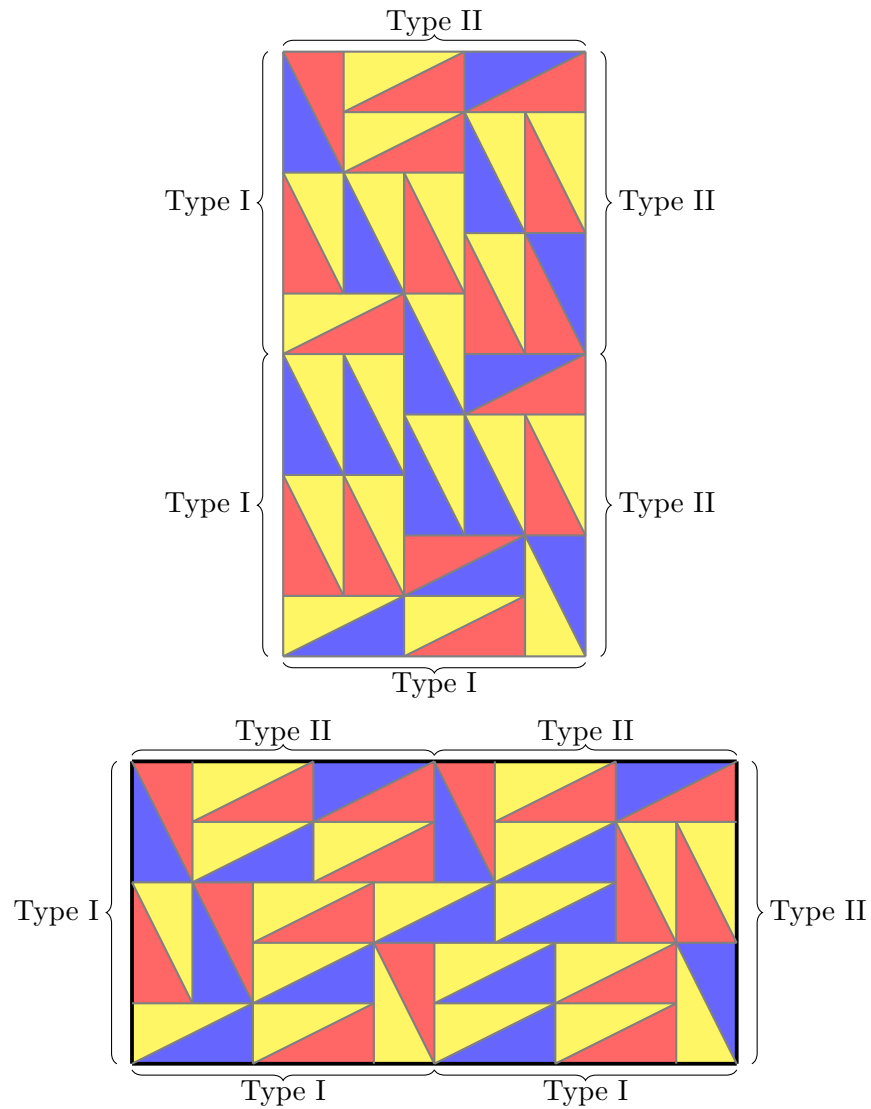
Type II

Type II

Type I

Type II

Type I

Type I

FIGURE 7. Face colouring of the level 2 supertiles in the rational pinwheel

a middle edge of a red triangle and a middle edge of a blue triangle while each Type II hypotenuse is made up in order exactly of a short edge of a red triangle, a middle edge of a yellow triangle and a middle edge of a blue triangle.

Next, since the hypotenuses of the tiles must coincide with the hypotenuses of other tiles, the same holds for the hypotenuses of the level 1 supertiles. This is the same fact we used in the proof of Thm. 5.2.

This means that every hypotenuse which is on the left or bottom edge of a rectangle (Type I) coincides with a hypotenuse on the right or top of a neighbouring rectangle (Type II) and every hypotenuse which is on the right or top edge of a rectangle (Type II) coincides with a hypotenuse on the left or bottom of a neighbouring rectangle (Type I).
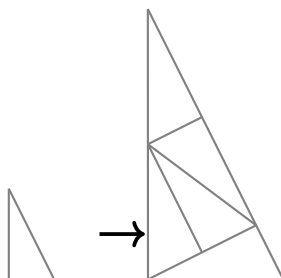
Therefore, we can only get conflicts along horizontal or vertical boundaries of Type I and Type II hypotenuses. In these two cases we get the following two scenarios, and there is clearly no conflict.
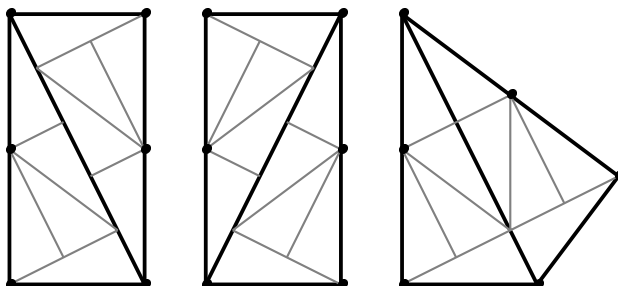


□
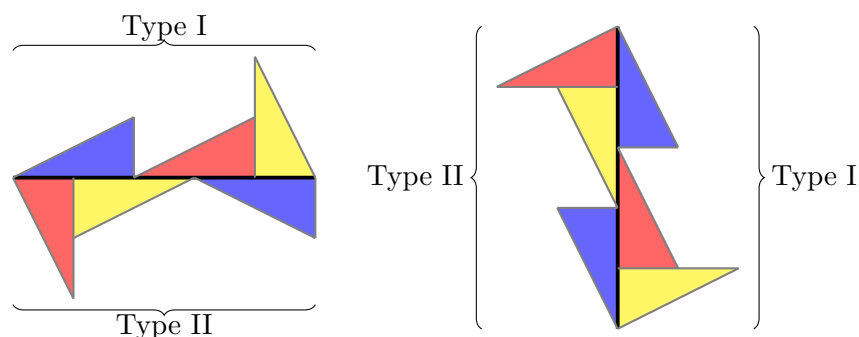
## 6. Pinwheel tiling

The pinwheel substitution is the single tile substitution:



For more details about this substitution, we recommend [1, 4, 8].

Let P denote the pinwheel tiling of the plane. The arguments here are similar to the rational pinwheel, but are a bit more complicated.

**Lemma 6.1.** *The tiles of P combine into the following three tiles and their rotations:*



*Furthermore, these tiles connect only at the nodes indicated.*

*Proof.* Similar to the arguments used in Lemma 5.1 and Thm. 5.2, the level one supertile can only combine hypotenuses to hypotenuses. Therefore, these three ways are the only possibilities up to rotation.

The last fact is straightforward since the boundaries of the three supertiles are entirely comprised of hypotenuses. Therefore, they must meet each other hypotenuse to hypotenuse and so the tiles connect only at the nodes indicated.                                                                        □

We can now calculate the chromatic number.

**Theorem 6.2.** *The pinwheel tiling has chromatic number $\chi_0(P) = 3$.*

*Proof.* An explicit 3-colouring of the pinwheel follows similarly to that of the rational pinwheel in Thm. 5.2.

By Lemma 6.1 we know that the pinwheel can be tiled by these three tiles and their rotations. Notice that their boundaries are 6-cycles and so the boundary subgraph of the pinwheel tiling is bipartite and can be 2-coloured with red and blue. All we then need to show is that we can colour the insides of these tiles with no conflicts:
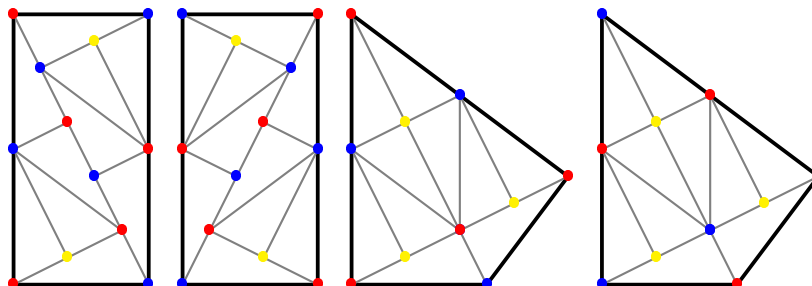


Notice that this accounts for all possible red/blue alternating boundary patterns since we can rotate the tiles.

Since $P$ contains 3-cycles, it is not bipartite. This proves $\chi(P) = 3$.     □

Distinct from the previous three sections, we cannot use a directional edge colouring since the pinwheel tiling irrationally rotates the tiles. We find the chromatic index via a different method.

**Lemma 6.3.** *The pinwheel tiling has $\Delta = 8$.*

*Proof.* The two irrational angles of a proto-tile sum to 180 degrees. Since one of these is the smallest interior angle of the triangle, the largest multiple of this angle is 4 since it must pair with 4 of the larger irrational angles to obtain a 360-degree angle sum around any vertex. Hence, the index cannot be more than 8.

To show that the index is exactly 8, notice that if we replace the isosceles triangle in the first level supertile
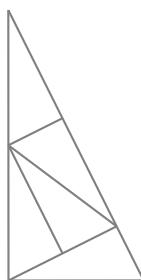


with first-level supertiles we get
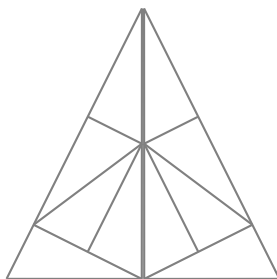


The central vertex has index 8.                                        $\square$

By Thm. 2.7 we know that the chromatic index of the pinwheel tiling is 8. However, we deduce a lot about the structure of the pinwheel tiling below and construct an explicit edge-colouring algorithm. This is of particular interest since we cannot use the Directional Alternating Colouring Method here.

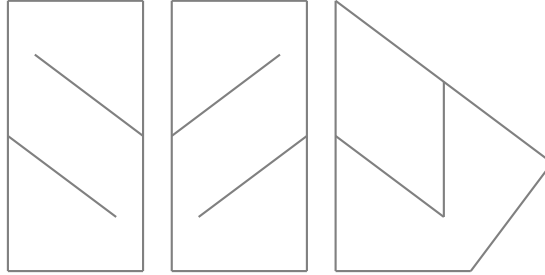We split the pinwheel tiling into two subgraphs with disjoint sets of edges:

$$P = P_{\sqrt{5}} \ \cup \ P_{1,2}$$

where $P_{\sqrt{5}}$ is the set of hypotenuses, or sides with length $\sqrt{5}$ and $P_{1,2}$ is the set of edges of length 1 or 2. Following immediately from Lemma 6.1 we get:

**Lemma 6.4.** *The $P_{1,2}$ is tiled by rotations of*



*and $P_{\sqrt{5}}$ is tiled by rotations of*



$\square$

Using the same argument from Lemma 6.3 we see that $P_{1,2}$ and $P_{\sqrt{5}}$ both have index 4. The goal is to four-colour each of these graphs.

Label the interior angles of the original $1, 2, \sqrt{5}$ triangle as $90, \alpha, \beta$, noting that the latter two angles are irrational.

Suppose $e$ and $f$ are two edges in $P_{1,2}$ (or equivalently $P_{\sqrt{5}}$) connected at a vertex. If we move from $e$ to $f$ the possible right-hand-side angle (or equivalently left-hand-side angle) is

$$0, \ 90, \ 180, \ 270, \ 2\alpha, \ 2\beta, \ 4\alpha, \ 4\beta, \ 2\alpha + 90, \ 2\beta + 90, \ 2\alpha + 180, \ 2\beta + 180$$

since $2\alpha + 2\beta = 180$ and are irrational angles. We call the move from $e$ to $f$ a **turn** if the angle is

$$90, \ 270, \ 2\alpha, \ 2\beta, \ 2\alpha + 180, \ 2\beta + 180$$

Define a relation on the edges of $P_{1,2}$, or equivalently $P_{\sqrt{5}}$, by the following: if $e, f$ are edges of $P_{1,2}$ then $e \sim f$ if and only if there is a path in $P_{1,2}$ with the first edge $e$, the last edge $f$ and the number of turns is even.

**Lemma 6.5.** *The relation on the edges of $P_{1,2}$, or $P_{\sqrt{5}}$ respectively, is an equivalence relation. Moreover, there are exactly two cosets.*

*Proof.* For any edge $e$ in $P_{1,2}$, we allow for the trivial path $e, e$ since the angle 0 is possible. Thus, $e \sim e$.

The relation is symmetric since $\gamma$ is the angle of a turn if and only if $360 - \gamma$ is as well. Thus, $e \sim f$ if and only if $f \sim e$.

Transitivity is easily verified and thus the relation is an equivalence relation.

Both $P_{1,2}$ and $P_{\sqrt{5}}$ are connected and so there are at most two cosets in each. Suppose we have a path $e_1, \ldots, e_k$ where $e_1 = e_k$, i.e. a cycle. This means that the right-hand-side angles must sum to an integer multiple of 180 which implies that there must be an even number of turns. Hence, all paths from $e$ to $f$ must contain an even number of turns or all paths from $e$ to $f$ must contain an odd number of turns. Therefore, there are exactly two cosets. $\qquad\square$

We can now prove the following lemma which will give us the chromatic index.

**Lemma 6.6.** *There are no simple cycles without turns in $P_{1,2}$ and $P_{\sqrt{5}}$, respectively.*

*Proof.* Assume for contradiction that a simple cycle without turns exists. Its interior (right-hand-side or left-hand-side) angles must only be from the list

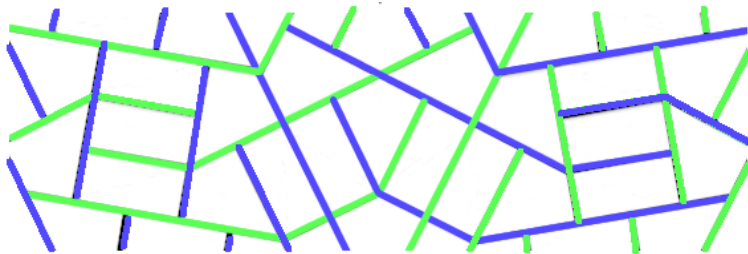$$180, \ 4\alpha, \ 4\beta, \ 2\alpha + 90, \ 2\beta + 90,$$

that is, neither turns nor 0. Any polygon must have an average interior angle strictly less than 180 degrees. Observe that

$$4\alpha + 4\beta = 2(180) = (2\alpha + 90) + (2\beta + 90), \quad \text{and}$$

$$4\alpha + 2(2\beta + 90) = 3(180) = 4\beta + 2(2\alpha + 90).$$

This, along with the irrationality of $\alpha$ and $\beta$, implies that the average interior angle is 180 degrees, which is a contradiction. $\qquad\square$

Here is a small picture of the two cosets of $P_{1,2}$:



We do not know if there are any infinite paths in a coset just that there aren't any cycles.

We can now provide an 8 edge-colouring for the pinwheel tiling.

**Theorem 6.7.** *The chromatic index of the pinwheel tiling is $\chi_1(P) = 8$.*

*Proof.* Lemma 6.3 says the chromatic index is at least 8. To show that there is an explicit 8 colouring is now possible.

Each coset of $P_{1,2}$ and $P_{\sqrt{5}}$ is a graph with no cycles and $\Delta = 2$. Thus, each coset is 2-colourable. Therefore, the pinwheel tiling is 8-colourable. $\qquad\square$

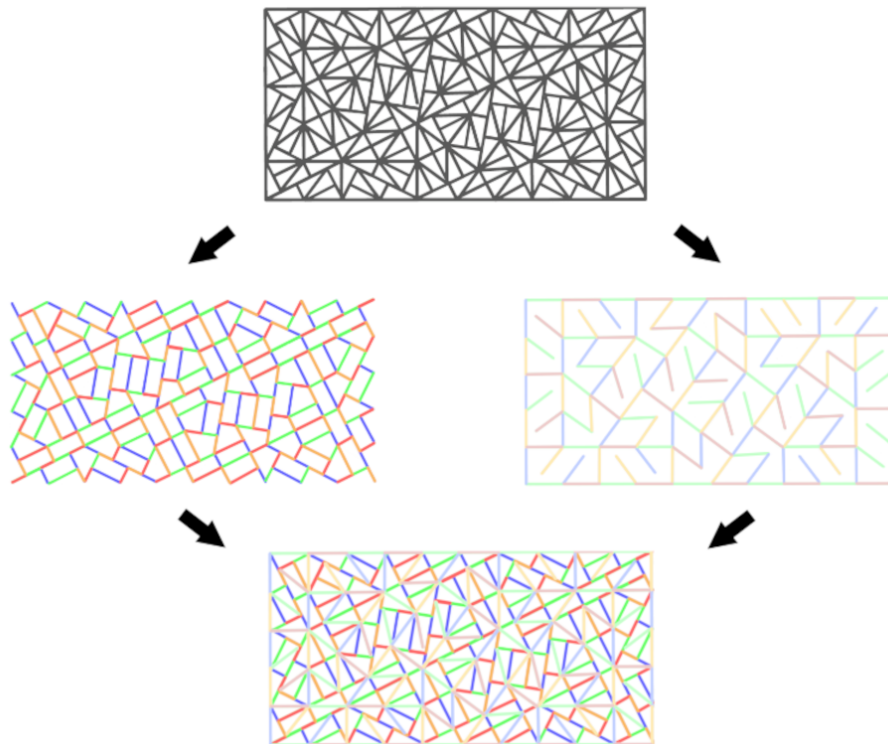*Remark 6.8:* A nonconstructive proof of Thm. 6.7 is given by Thm. 2.7.



FIGURE 8. Colouring of edges in the pinwheel tiling.

Lastly we turn to the face colouring of the pinwheel tiling.

**Theorem 6.9.** *The pinwheel tiling satisfies $\chi_2(P) = 3$.*

*Proof.* Lemma 6.3 and Thm. 6.2 prove that the pinwheel tiling can be tiled by rectangles and kites whose boundary vertices are 2-colourable. Label this two colouring with A's and B's. Finally, colour the fourth-level supertiles as in Figure 9, matching vertex labels to the two colouring of the boundaries.
□

A larger patch of the pinwheel tiling which helps better understand the 3-colouring procedure is included in Figure 10.
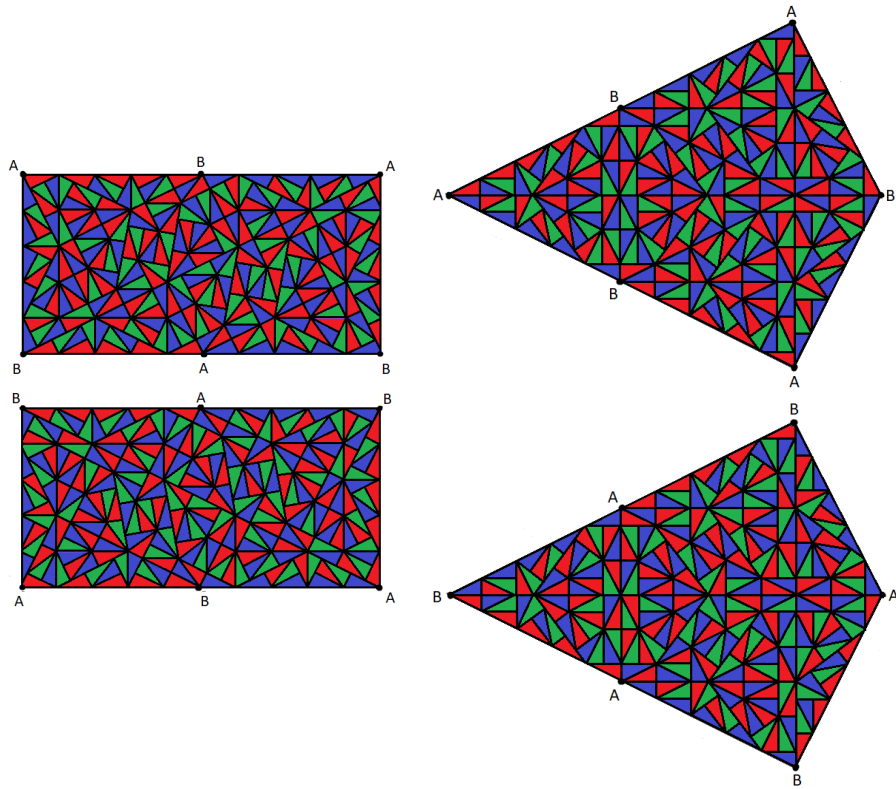
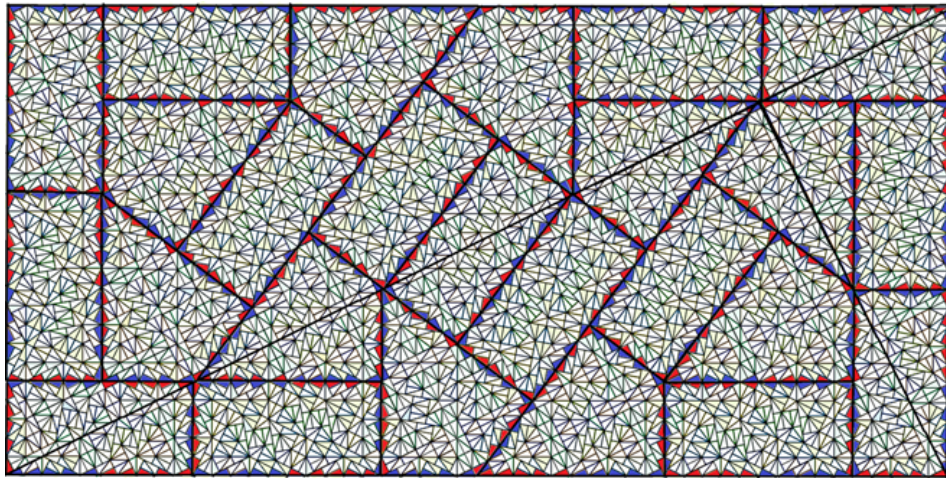FIGURE 9. Face colouring of the supertiles



FIGURE 10. Face colouring around boundaries of supertiles

## References

1. M. Baake and U. Grimm, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge University Press, 2013.

2. M. Baake, R. V. Moody, and Schlottmann M., *Limit-(quasi) periodic point sets as quasicrystals with p-adic internal spaces*, J. Phys. A: Mathematical and General **31** (1998), no. 27, 55–65.

3. F. P. M. Beenker, *Algebraic theory of non-periodic tilings of the plane by two simple building blocks: a square and a rhombus*, vol. 82-WSK04, Eindhoven University of Technology, 1982.

4. D. Frettlöh, E. Harriss, and F. Gähler, *Tilings encyclopedia*, `https://tilings.math.uni-bielefeld.de/`.

5. C. Goodman-Strauss, *A small aperiodic set of planar tiles*, European J. Combin. **20** (1999), 375–384.

6. W. H. Gottschalk, *Choice functions and Tychonoff's theorem*, Proc. AMS **2** (1951), no. 1, 172.

7. I. Holyer, *The NP-completeness of edge-coloring*, SIAM Jour. Comput. **10** (1981), no. 4, 718–720.

8. C. Radin, *The pinwheel tilings of the plane*, Ann. Math **139** (1994), 661–702.

9. D. P. Sanders and Y. Zhao, *Planar graphs of maximum degree seven are class I*, J. Combin. Th., Series B **83** (2001), no. 2, 201–212.

10. T. Sibley and S. Wagon, *Rhombic Penrose tilings can be 3-colored*, Amer. Math. Mont. **107** (2000), 251–253.

11. V. G. Vizing, *Critical graphs with given chromatic class*, Metody Diskret. Analiz. **5** (1965), 9–17.

12. R. J. Wilson, *Introduction to Graph Theory*, Prentice Hall, Essex, 2010.

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