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LINEAR ARBORICITY OF THE TENSOR PRODUCTS OF COMPLETE MULTIPARTITE GRAPHS

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Abstract. The *linear arboricity* of a graph G, denoted by $\ell a(G)$, is the minimum number of linear forests which partition the edge set of G. Akiyama et al. conjectured that $\ell a(G) = \lceil \frac{k+1}{2} \rceil$ for any k-regular graph G. This conjecture is proved to be true for $k = 3, 4, 5, 6, 8, 10$. Also, in [P. Paulraja and S. Sivasankar, Linear Arboricity of the Tensor Products of Graphs, *Utilitas Math.* 99 (2016) 295–317], the conjecture is proved for the tensor product of complete graphs. Although the conjecture was not proved in general, we have proved that the tensor product of two regular complete multipartite graphs confirms the conjecture in the affirmative.

1. Introduction

All graphs considered here are simple and finite. Let C_k (resp. P_k) denote the cycle (resp. path) on k vertices. Let K_m denote the complete graph. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the *subgraph of G induced by S*. Similarly, for $F \subseteq E(G)$, $\langle F \rangle$ denotes the *edge induced subgraph of* G induced by F. For a graph G, if its edge set $E(G)$ can be partitioned into $E_1, E_2, \dots, E_k$ such that $\langle E_i \rangle \cong H$, for all $i, 1 \leq i \leq k$, then we say that H *decomposes* G. A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of G is a spanning subgraph of G in which each vertex is of degree k. A *k-factorization* $\mathcal{F} = \{F_1, F_2, \dots, F_\ell\}$ of the graph G is a partition of $E(G)$ into $F_1, F_2, \dots, F_\ell$, where $\langle F_i \rangle, 1 \leq i \leq \ell$, is a k-factor of G. A subgraph H of G is *orthogonal* to a k-factorization $\mathcal{F} = \{F_1, F_2, \dots, F_\ell\}$ of G if $|E(H) \cap E(F_i)| = 1, 1 \leq i \leq \ell$.

A k-regular graph G is called *Hamilton cycle decomposable* if G is decomposable into $\frac{k}{2}$ Hamilton cycles when k is even and into $(k-1)/2$ Hamilton cycles together with a 1-factor (perfect matching) when k is odd. For a real number x, $\lceil x \rceil$ denotes the least integer not less than x. A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, 3, \dots, n\}$, such that each row and each column of the array

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contains each of the symbols in $\{1, 2, \ldots, n\}$ exactly once. A *transversal* T of a latin square of order n on the symbols $\{1, 2, \ldots, n\}$ is a set of n cells, exactly one cell from each row and each column, such that each of the symbols in $\{1, 2, \ldots, n\}$ occurs in a cell of T.

For two simple graphs G and H their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H, see Figure 1. Similarly, the *wreath product* of the graphs G and H, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G, or $g_1 = g_2$ and h_1h_2 is an edge in H. Note that $K_m \circ \overline{K}_n$ is the complete m partite graph in which each partite set has n vertices. If $H_1, H_2, \ldots, H_k$ are edge disjoint subgraphs of G which partition the edges of G, then we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. It is known [14] that the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$.

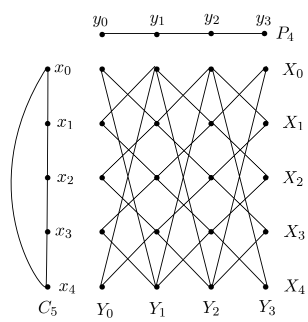


FIGURE 1. The graph $C_5 \times P_4$.

We shall use the following notation throughout this paper. Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. If G contains the set of edges $F_i(X, Y) = \{x_jy_{j+i} \mid 0 \le j \le n-1$, where subscripts are taken modulo $n\}$, $0 \le i \le n-1$, then we say that G has the *1-factor of jump i from X to Y*. Note that $F_i(X, Y) = F_{n-i}(Y, X), 0 \le i \le n-1$, where the suffix is taken modulo n and we assume $F_n(X, Y) = F_0(X, Y) = F_0(Y, X)$. Clearly, if $G = K_{n,n}$, then $E(G) = \cup_{i=0}^{n-1} F_i(X, Y)$.

Let G and H be simple graphs with vertex sets $V(G) = \{x_0, x_1, \ldots, x_{m-1}\}$ and $V(H) = \{y_0, y_1, \ldots, y_{n-1}\}$, then $V(G \times H) = V(G) \times V(H)$ and for our convenience, we write $V(G) \times V(H) = \cup_{i=0}^{m-1} X_i$, where X_i stands for $\{x_i\} \times V(H)$. In what follows, we shall denote the set of vertices of X_i, $0 \le i \le m-1$, by $\{x_{i,j} \mid 0 \le j \le n-1\}$, where $x_{i,j}$ stands for the vertex (x_i, y_j) of $G \times H$. We shall call X_i, the *ith layer of* $G \times H$. Further, we shall call $Y_j = \{x_{i,j} \mid 0 \le i \le m-1\}$, $0 \le j \le n-1$, the *jth column of* $G \times H$, see Figure 1. It is clear that $G \times H$ is an m-partite graph with partite sets $X_0, X_1, \ldots, X_{m-1}$; it can also be considered as an n-partite graph with partite sets $Y_0, Y_1, \ldots, Y_{n-1}$, where $Y_i = V(G) \times \{y_i\}$. Definitions not seen here can be found in [2] or [3].

A *linear forest* is a graph in which each of its components is a path. The *linear arboricity* of a graph G, denoted by $\ell a(G)$, as defined by Harary [7], is the minimum number of linear forests which partition the edge set of G. Akiyama et al. [8] posed the following conjecture.

Conjecture 1 ([8])**.** *For any k-regular graph $G, \ell a(G) = \left\lceil \frac{k+1}{2} \right\rceil$.*

The above conjecture is proved for complete graphs and graphs with maximum degree $3, 4, 5, 6, 8, 10$; see [9, 5, 6, 8]. The above conjecture is equivalent to the following conjecture, which we call the linear arboricity conjecture (LAC).

Conjecture 2 (LAC)**.** *For any graph G, $\left\lceil \frac{\Delta(G)}{2} \right\rceil \leq \ell a(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

LAC has been verified to be true for the classes of graphs such as complete bipartite graphs, trees, Halin graphs, series-parallel graphs and planar graphs, see [8, 15, 16, 17, 19]. Alon [1] proved that for every $\epsilon > 0$, $\ell a(G) \leq (\frac{1}{2} + \epsilon)\Delta$ for every graph G with sufficiently large Δ.

Muthusamy and Paulraja [11] and independently Wu [18] verified the LAC to be true for regular complete multipartite graphs $K_r \circ \overline{K}_s, r \geq 3$. In [13], LAC has been verified to be true for the graphs $K_r \times K_s, r, s \geq 2$, and $K_{r,r} \times K_s, r \geq 1, s \geq 2$. In this paper, we consider the linear arboricity of the tensor product of regular complete multipartite graphs.

We prove the following theorem in support of Conjecture 2 using the results of [13].

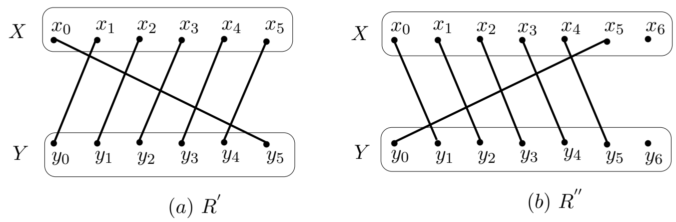
Theorem 1.1. *For $m, n, r, s \geq 3$, the $\ell a\left((K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)\right)$ is*

$$\left\lceil \frac{\Delta\left((K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)\right) + 1}{2} \right\rceil = \left\lceil \frac{(r-1)(m-1)ns + 1}{2} \right\rceil.$$

2. Main Result

In this section, we prove that the LAC is true for the graph $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)$. Paulraja and Sivasankar proved that the LAC is true for the tensor product of complete graphs $K_r \times K_s, r, s \geq 2$, see [13]. They achieved this by considering three different cases, namely (i) r and s are even, (ii) r is even and s is odd and (iii) both r and s are odd. The following are the notation and results in [13], which we use extensively in the proof of our main theorem.

First we define two graphs, R' and R'', which are subgraphs of $K_{2r,2r} - F_0(X,Y)$ and $K_{2r+1,2r+1} - F_0(X,Y)$, respectively, as follows: let $X = \{x_0, x_1, \ldots, x_{2r-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{2r-1}\}$. The subgraph $R' = \langle F_{2r-1}(X,Y) \rangle$, where $F_{2r-1}(X,Y)$ is the 1-factor of jump $2r-1$ from X to Y in $K_{2r,2r}$, see Figure 2(a). Similarly, $R'' = \{\langle F_1(X,Y) \rangle - \{x_{2r-1}y_{2r}, x_{2r}y_0\}\} \cup x_{2r-1}y_0$, where $X = \{x_0, x_1, \ldots, x_{2r}\}$ and



(a) The subgraph $R^{'}$ of $K_{6,6} - F_0(X, Y)$.
(b) The subgraph $R^{''}$ of $K_{7,7} - F_0(X, Y)$.

FIGURE 2. Graphs R' and R'' when $r = 3$.

$Y = \{y_0, y_1, \ldots, y_{2r}\}$, see Figures 2(a) and 2(b) for R' and R'', respectively, when $r = 3$.

We list below some known results for our future reference.

Lemma 2.1 ([13])**.** *Let the cycle $C_{2r} = x_0x_1x_2 \ldots x_{2r-1}x_0$. For $r \geq 2$ and $m \geq 1$, $C_{2r} \times K_{2m}$ can be decomposed into $2(m-1)$ Hamilton paths, a linear forest and a matching R' (defined above), which is contained in $\langle X_{2r-1} \cup X_0 \rangle \subseteq C_{2r} \times K_{2m}$; that is, linear arboricity conjecture is true for $C_{2r} \times K_{2m}$.* □

The linear forest and R' of Lemma 2.1 are shown in Figure 3 for $2r = 6$ and $2m = 8$.

Lemma 2.2 ([13])**.** *For $r \geq 2$ and $m \geq 1$,*

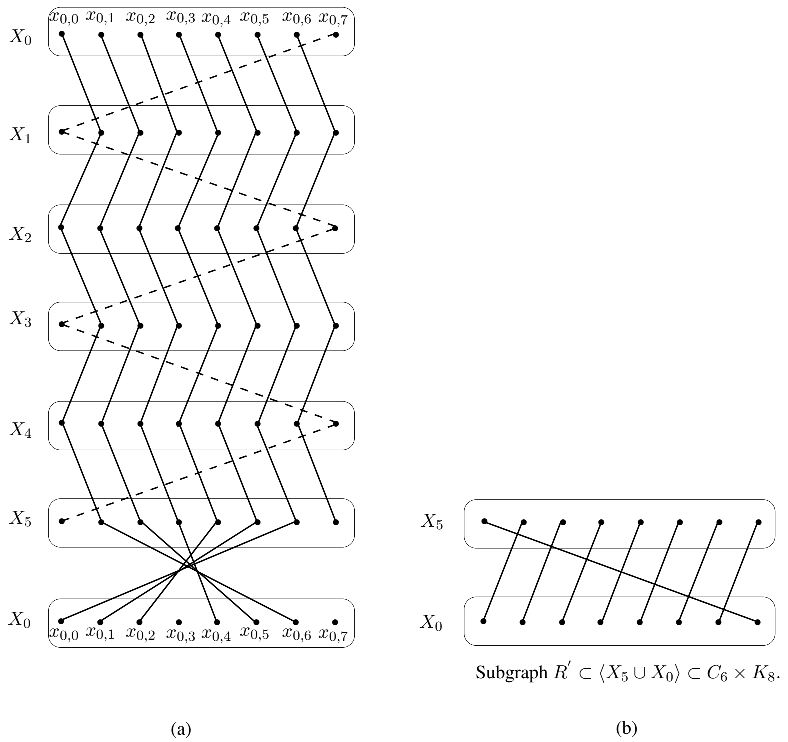
$$\ell a(K_{2r} \times K_{2m}) = \left\lceil \frac{(2r-1)(2m-1)+1}{2} \right\rceil.$$

Lemma 2.3 ([13])**.** *Let $C_r = x_0x_1 \ldots x_{r-1}x_0$. For $r \geq 3$ and $m \geq 1$, the graph $C_r \times K_{2m+1}$ is decomposable into $2m$ Hamilton paths and a matching R'' of $\langle X_{r-1} \cup X_0 \rangle \subset C_r \times K_{2m+1}$; that is, the linear arboricity conjecture is true for $C_r \times K_{2m+1}$.* □

Lemma 2.4 ([13])**.** *For $r, m \geq 1, \ell a(K_{2r} \times K_{2m+1}) = \left\lceil \frac{(2r-1)2m+1}{2} \right\rceil$.*

Lemma 2.5 ([13])**.** *For $r, m \geq 1, \ell a(K_{2r+1} \times K_{2m+1}) = \left\lceil \frac{4rm+1}{2} \right\rceil$.* □

Clearly, $K_r \circ \overline{K}_s$ (resp. $K_m \circ \overline{K}_n$) is the complete multipartite graph with partite sets $X_0, X_1, X_2, \ldots, X_{r-1}$ (resp. $Y_0, Y_1, Y_2, \ldots, Y_{m-1}$), where each partite set has s vertices (resp. n vertices). Consider the tensor product of two regular complete multipartite graphs, that is, $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n), m, n, r, s \geq 3$. Let $X_{i,j}$ denote the set $X_i \times Y_j \subset V((K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n))$. If we identify each independent set $X_{i,j}$, of size ns, as a vertex $x_{i,j}$ and join $x_{i,j}$ and $x_{k,\ell}$ by an edge if and only if $\langle X_{i,j} \cup X_{k,\ell} \rangle$ is a complete bipartite graph $K_{ns,ns}$, then the resulting graph is isomorphic to the graph $K_r \times K_m$, see Figure 4. Hence the graph $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n) \cong (K_r \times K_m) \circ \overline{K}_{ns}$.



(a) A linear forest in $C_6 \times K_8$ obtained by the edges deleted from the six Hamilton cycles of $C_6 \times K_8$ (as in Lemma 2.1) together with

$$\{\cup_{i=0}^{2}\{F_1(X_{2i}, X_{2i+1}) \cup F_7(X_{2i+1}, X_{2i+2})\}\} - F_7(X_5, X_0)$$

in $C_6 \times K_8$. The broken and solid edges constitute two paths in this linear forest. For clarity X_0 is drawn twice; if the last X_0 when superimposed with the first X_0 we get the desired paths.
(b) The subgraph R' of $\langle X_5 \cup X_0 \rangle$, contained in $C_6 \times K_8$, is obtained after the deletion of the edges of the 6 Hamilton paths and a linear forest described in (a).

Figure 3. A linear forest and R' in $C_6 \times K_8$.

This means that corresponding to each edge of $K_r \times K_m$, we have a complete bipartite graph $K_{ns,ns}$ in $(K_r \times K_m) \circ \overline{K}_{ns}$, see Figure 4.

The main idea and outline of the proof of Theorem 1.1 is the following. We prove that the LAC is true for the graph $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)$ in three cases: (i) r and m are even (ii) r is even and m is odd and (iii) r and m are odd. In all three cases, we reduce the graph $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)$ into $K_r \times K_m$ by identifying a set of vertices as described in Figure 4. We use the linear forest decomposition of $K_r \times K_m$ in [13] and "blow" up each of the linear forests into the corresponding subgraph of $(K_r \times K_m) \circ \overline{K}_{ns}$ to obtain the required number of linear forests in the graph $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)$. During the process of blowing up the vertices, we need the linear forest decomposition of some special graphs described in this paper, where we

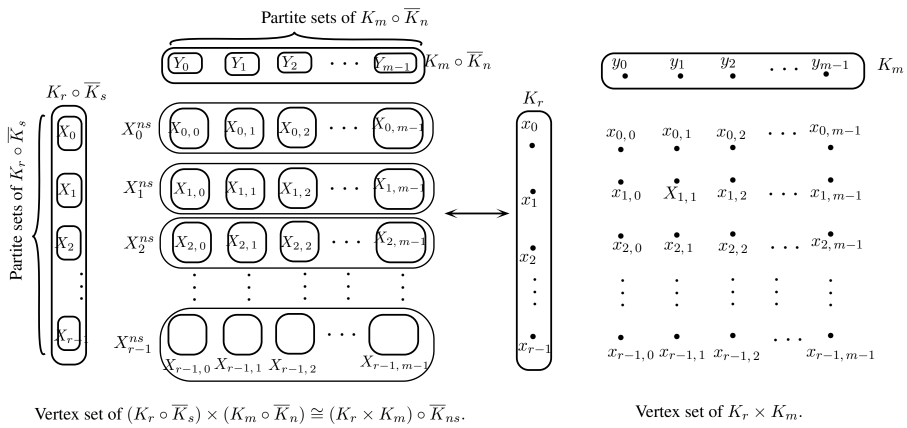


FIGURE 4. A small box $X_{i,j}$ denotes the ns independent vertices $X_i \times Y_j$. The vertex $x_{i,j}$ of $K_r \times K_m$ is the corresponding vertex of the independent set $X_{i,j}$ of size ns in $(K_r \times K_m) \circ \overline{K}_{ns}$ and vice versa. Also the layers of $(K_r \times K_m) \circ \overline{K}_{ns}$ are denoted by $X_i^{ns}, 0 \leq i \leq r-1$.

have identified perfect matchings or partial matchings to concatenate the vertices to make the requiered number of linear trees.

The following theorem is proved in [4]. For the sake of completeness we give the proof of it as we need the decomposition of the graph for our future reference.

Theorem 2.6. *For $r \geq 3$ and $s \neq 2, 6$, $C_r \circ \overline{K}_s$ can be decomposed into Hamilton cycles.*

Proof. Let $C_r = 1\,2\,3 \ldots (r-1)\,r\,1$. Since $s \neq 2, 6$, there exists a pair of orthogonal latin squares, of order s, say L_s and L'_s with symbol set $\{0, 1, 2, \ldots, s-1\}$, see [10]. Then L_s can be partitioned into s disjoint transversals, see [10]. We denote the elements of L_s by ordered triples (α, β, γ), where α is the row index, β is the column index and γ is the symbol in cell (α, β). Let $T_1, T_2, \ldots, T_s$ be the s disjoint transversals of L_s. To each T_i we construct a Hamilton cycle in $C_r \circ \overline{K}_s$ as follows. If the αth row element of T_i is the triple (α, β, γ), then we construct a path $P_i^{\alpha}, 0 \leq \alpha \leq s-1$, in $C_r \circ \overline{K}_s$ as follows: $P_i^{\alpha} = (1, \alpha)(2, \beta)(3, \alpha)(4, \beta) \ldots (r-1, \alpha)(r, \beta)$, if r is even and $(1, \alpha)(2, \beta)(3, \alpha)(4, \beta) \ldots (r-1, \beta)(r, \gamma)$, if r is odd. The ith Hamilton cycle is obtained by the union of s paths $P_i^0, P_i^1, \ldots, P_i^{s-1}$ and then join an edge from the last vertex of each $P_i^j, 0 \leq j \leq s-1$, to the first vertex of P_i^{j+1}, where the addition is taken modulo s. Let E_i be the set of edges that have been added to get the ith Hamilton cycle, then the ith Hamilton cycle is $H_i = P_i^1 \cup P_i^2 \cup \ldots \cup P_i^s \cup E_i$. Proceeding like this for every transversal, we get a Hamilton cycle decomposition of $C_r \circ \overline{K}_s$. $E_i \cap E_j = \phi, i \neq j$, can be verified by the latin square property.

This completes the proof of the theorem. □

Lemma 2.7. *For $r \geq 3$ and $s \neq 2, 6$, $C_r \circ \overline{K}_s$ admits a Hamilton cycle decomposition $\mathscr{H}$ such that for any given fixed j, there is a 1-factor in $\langle X_j \cup X_{j+1} \rangle$ which is orthogonal to $\mathscr{H}$.*

Proof. Let $C_r = 1\,2\,3 \dots (r-1)\,r\,1$. Fix the jth edge $j(j+1)$ of C_r, where addition is taken modulo r with residues $1, 2, \dots, r$. Let $\mathscr{H} = \{H_i | 1 \leq i \leq s\}$ be a Hamilton cycle decomposition of $C_r \circ \overline{K}_s$, by Theorem 2.6. Let us denote by M the set of jth edge of each of the paths $P_1^1,\ P_2^2,\ P_3^3,\ \dots,\ P_{s-1}^{s-1},\ P_s^s$ contained in the Hamilton cycles $H_1, H_2, \dots, H_s$, respectively, see the proof of Theorem 2.6. Clearly, M is a 1-factor in the subgraph induced by the jth and $j+1$th layers of $C_r \circ \overline{K}_s$, and is orthogonal to the Hamilton cycle decomposition $\mathscr{H}$ of $C_r \circ \overline{K}_s$, since $P_i^i, 1 \leq i \leq s$, is a section of the Hamilton cycle $H_i, 1 \leq i \leq s$. □

For the rest of the paper, in the graph $G \times H$, irrespective of the nature of the graphs G and H, the vertex $x_{i,j}$ denotes the ordered pair $(x_i, y_j) \in V(G) \times V(H)$. If $x_{i,j} \in V(G \times H)$, then $X_{i,j}$ denotes the set of vertices in $(G \times H) \circ \overline{K}_{ns}$ obtained by replacing $x_{i,j}$ by ns independent vertices.

Lemma 2.8. *For $r, m \geq 2$ and $ns \neq 2, 6$, the graph $(C_{2r} \circ \overline{K}_s) \times (K_{2m} \circ \overline{K}_n)$ can be decomposed into $(2m-2)ns$ Hamilton paths, ns linear forests and a matching, that is, the LAC is true for $(C_{2r} \circ \overline{K}_s) \times (K_{2m} \circ \overline{K}_n)$.*

Proof. Let $C_{2r} = x_0 x_1 x_2 \dots x_{2r-1} x_0$. If $n = s = 1$, then the result follows from Lemma 2.1 and hence we assume that $n, s > 1$. It is an easy observation that $(C_{2r} \circ \overline{K}_s) \times (K_{2m} \circ \overline{K}_n) \cong (C_{2r} \times K_{2m}) \circ \overline{K}_{ns}$. First we decompose $C_{2r} \times K_{2m}$ into Hamilton cycles $H_1, H_2, \dots, H_{2(m-1)}$ and a 2-factor G, see [13]. Then we can write $(C_{2r} \times K_{2m}) \circ \overline{K}_{ns} = \left(\bigoplus_{i=1}^{2(m-1)} (H_i \circ \overline{K}_{ns})\right) \oplus (G \circ \overline{K}_{ns})$. Now we decompose each $H_i \circ \overline{K}_{ns}, 1 \leq i \leq 2(m-1)$ and $G \circ \overline{K}_{ns}$ into appropriate number of linear forests.

The $2(m-1)$ Hamilton cycles, $H_j, 1 \leq j \leq 2(m-1)$ of $C_{2r} \times K_{2m}$ are

$$H_{2i-1} = \bigcup_{t \text{even}} \{\{F_{2m-2i}(X_t, X_{t+1}) \cup F_{2m-2i+1}(X_t, X_{t+1})\} - x_{t,m+i-1}x_{t+1,m-i}\} \cup \bigcup_{t \text{odd}} \{x_{t,m-i}x_{t+1,m+i-1}\},$$

$$H_{2i} = \bigcup_{t \text{odd}} \{\{F_{2i-1}(X_t, X_{t+1}) \cup F_{2i}(X_t, X_{t+1})\} - x_{t,m-i}x_{t+1,m+i-1}\} \cup \bigcup_{t \text{even}} \{x_{t,m+i-1}x_{t+1,m-i}\},$$

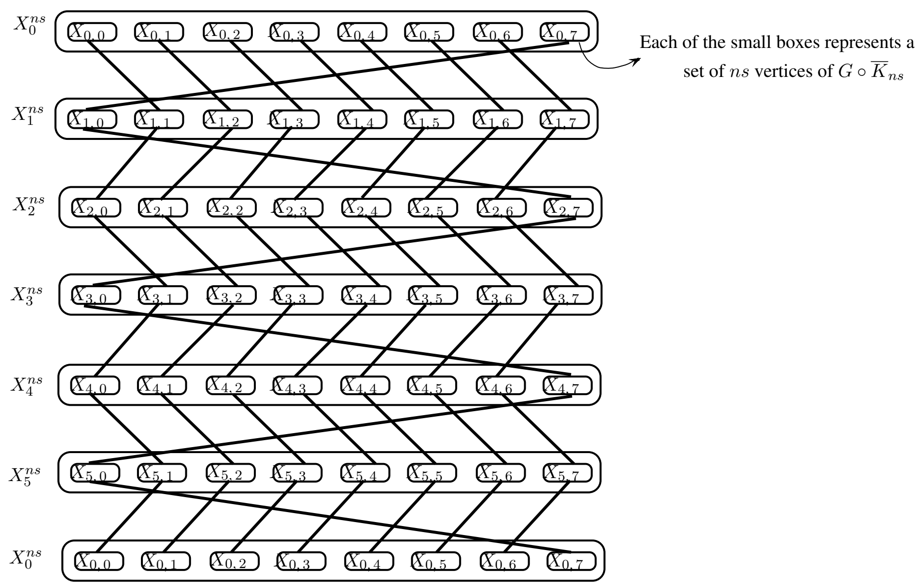
for $0 \leq t \leq 2r-1, 1 \leq i \leq m-1$, and the last 2-factor is

$$G = \bigcup_{i=0}^{r-1} F_1(X_{2i}, X_{2i+1}) \cup \bigcup_{i=0}^{r-1} F_{2m-1}(X_{2i+1}, X_{2i+2}),$$

where the addition in the subscripts of X is taken modulo $2r$.

Let

$$R = \{x_{2r-1,m-1}x_{0,m}, x_{2r-1,m-2}x_{0,m+1}, x_{2r-1,m-3}x_{0,m+2}, \ldots, x_{2r-1,1}x_{0,2m-2}, x_{2r-1,2m-2}x_{0,0}, x_{2r-1,2m-3}x_{0,1}, x_{2r-1,2m-4}x_{0,2}, \ldots, x_{2r-1,m}x_{0,m-2}\}.$$



A bold line represents a $K_{ns,ns}$ between the respective vertex subsets. In this figure, the layer X_0^{ns} of $G \circ \overline{K}_{ns}$ appears twice. Identifying the first and last X_0^{ns}, we get the graph isomorphic to $G \circ \overline{K}_{ns}$.

FIGURE 5. $G \circ \overline{K}_{ns}$.

R is orthogonal to the above set of $2(m-1)$ Hamilton cycles $H_i, 1 \leq i \leq 2(m-1)$ since $|H_i \cap R| = 1$ and $H_{2i-1} \cap R = x_{2r-1,m-i}x_{0,m+i-1}, 1 \leq i \leq m-1$ and $H_{2i} \cap R = x_{2r-1,2m-i-1}x_{0,i-1}, 1 \leq i \leq m-1$, see the proof of Lemma 2.7 of [13].

Thus $(C_{2r} \times K_{2m}) \circ \overline{K}_{ns} = (H_1 \oplus H_2 \oplus \ldots \oplus H_{2(m-1)} \oplus G) \circ \overline{K}_{ns} = (H_1 \circ \overline{K}_{ns}) \oplus (H_2 \circ \overline{K}_{ns}) \oplus \ldots \oplus (H_{2(m-1)} \circ \overline{K}_{ns}) \oplus (G \circ \overline{K}_{ns})$, where $\{H_i | 1 \leq i \leq 2(m-1)\}$ is the Hamilton cycle decomposition of $C_{2r} \times K_{2m}$ obtained above, and G is a 2-factor of $C_{2r} \times K_{2m}$ described above.

In the graph $H_{2i-1}, 1 \leq i \leq m-1$, fix the edge $x_{2r-1,m-i}x_{0,m+i-1}$; then $H_{2i-1} \circ \overline{K}_{ns}$ admits a Hamilton cycle decomposition $\{H_{2i-1}^j | 1 \leq j \leq ns\}$ such that a perfect matching M_{2i-1} in $\langle X_{2r-1,m-i} \cup X_{0,m+i-1} \rangle$ is orthogonal to the cycles $\{H_{2i-1}^j | 1 \leq j \leq ns\}$, by Lemma 2.7, where $X_{i,j} = x_{i,j} \times V(K_{ns})$ and $x_{i,j} \in V(C_{2r} \times K_{2m})$. Thus $H_{2i-1} \circ \overline{K}_{ns} = \bigoplus_{j=1}^{ns} H_{2i-1}^j = \bigoplus_{j=1}^{ns} P_{2i-1}^j \oplus M_{2i-1}$, where P_{2i-1}^j is the Hamilton path obtained from the Hamilton cycle H_{2i-1}^j by deleting the edge of M_{2i-1} in it. Similarly, in the graph $H_{2i}, 1 \leq i \leq m-1$, fix the edge $x_{2r-1,2m-i-1}x_{0,i-1}$; then $H_{2i} \circ \overline{K}_{ns}$ admits a Hamilton cycle decomposition $\{H_{2i}^j | 1 \leq j \leq ns\}$ such that a perfect matching M_{2i} in $\langle X_{2r-1,2m-i-1} \cup X_{0,i-1}\rangle$ is orthogonal to the cycles $\{H_{2i}^j | 1 \leq j \leq ns\}$, by Lemma 2.7. Thus $H_{2i} \circ \overline{K}_{ns} = \bigoplus_{j=1}^{ns} H_{2i}^j = \bigoplus_{j=1}^{ns} P_{2i}^j \oplus M_{2i}$, where P_{2i}^j is the Hamilton path obtained from the Hamilton cycle H_{2i}^j by deleting the edge of M_{2i} in it. We denote $\bigcup_{i=1}^{2(m-1)} M_i$, by M; M is a matching in $\langle X_{2r-1}^{ns} \cup X_0^{ns}\rangle \subset (C_{2r} \times K_{2m}) \circ \overline{K}ns$.

Consider the 2-factor G of $C_{2r} \times K_{2m}$, obtained at the beginning of this lemma, which consists of $2m$ cycles $C_{2r}^i, 1 \leq i \leq 2m$, each of length $2r$. Now $G \circ \overline{K}_{ns} = \bigoplus_{i=1}^{2m}(C_{2r}^i \circ \overline{K}_{ns})$, see Figure 5 for $2r = 6$ and $2m = 8$. Fix the edge $x_{2r-1,i+1}x_{0,i}$ of $C_{2r}^i, 1 \leq i \leq 2m$; then each $C_{2r}^i \circ \overline{K}_{ns}$ has a Hamilton cycle decomposition such that a matching M_i' in $\langle X_{2r-1,i+1} \cup X_{0,i}\rangle$ is orthogonal to the Hamilton cycle decomposition of $C_{2r}^i \circ \overline{K}_{ns}$, by Lemma 2.7. Thus each $C_{2r}^i \circ \overline{K}_{ns}, 1 \leq i \leq 2m$, has a decomposition into ns Hamilton paths and a matching. Let $M' = \bigcup_{i=1}^{2m} M_i'$. M' is a 1-factor of $\langle X_{2r-1}^{ns} \cup X_0^{ns}\rangle$. Thus we have a decomposition of $(G \circ \overline{K}_{ns}) - M'$ into paths, where each path has $2rns$ vertices. Thus we have a P_{2rns}-decomposition of $(G \circ \overline{K}_{ns}) - M'$.

M and M' are matchings in $\langle X_{2r-1}^{ns} \cup X_0^{ns}\rangle$, see the broken lines of Figures 6(a) and 6(b), respectively, for $2r = 6$ and $2m = 8$. The linear forest decomposition of $((G \circ \overline{K}_{ns}) - M') \cup M$, consisting of ns linear forests, is obtained as follows: As G consists of $2m$ copies of C_{2r}, collecting one P_{2rns} appropriately from each component of $(G \circ \overline{K}_{ns}) - M'$ together with some edges of the matching M connecting the ends of P_{2rns} constitute a linear forest of $\big(((G \circ \overline{K}_{ns}) - M') \cup M\big) \subset (C_{2r} \times K_{2m}) \circ \overline{K}_{ns}$. In otherwords, the Hamilton path decomposition of the components of $(G \circ \overline{K}_{ns}) - M'$ (that is, the P_{2rns}-decomposition obtained above) when fused with the edges of M, we get ns linear forests of $((G \circ \overline{K}_{ns}) - M') \cup M$. Thus we have decomposed $((G \circ \overline{K}_{ns}) - M') \cup M$ into ns linear forests.

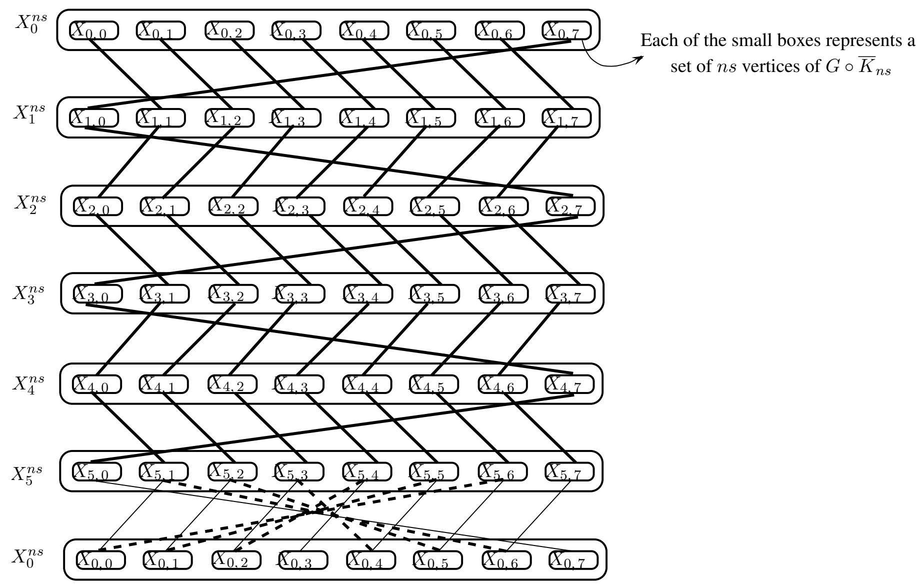
Thus we have decomposed $(C_{2r} \times K_{2m}) \circ \overline{K}_{ns}$ into $2(m-1)ns$ Hamilton paths, ns linear forests, distinct from the Hamilton paths, and a matching M' and hence $\ell a((C_{2r} \times K_{2m}) \circ \overline{K}_{ns}) = 2(m-1)ns + ns + 1 = 2mns - ns + 1$.

This completes the proof of the lemma. □

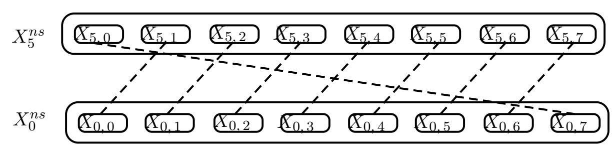
To prove Lemma 2.10, we define a graph P_{2r}^ℓ and show that LAC is true for P_{2r}^ℓ.

The graph P_{2r}^ℓ is obtained from the path $P_{2r} = x_0x_1 \ldots x_{2r-1}$, by replacing each vertex x_i in P_{2r} by the set of ℓ independent vertices $X_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,\ell-1}\}$. The edge set of P_{2r}^ℓ consists of the following edges: for $0 \leq i \leq r-1$, $\langle X_{2i} \cup X_{2i+1}\rangle$ induces a $K_{\ell,\ell}$ and for $0 \leq i \leq r-2$, $\langle X_{2i+1} \cup X_{2i+2}\rangle$ is a 1-factor $F_0(X_{2i+1}, X_{2i+2})$, see Figure 7(a).

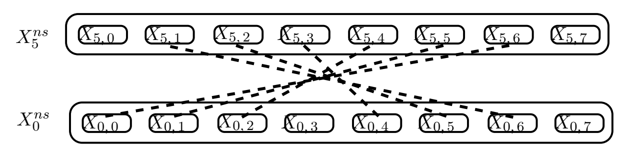
Lemma 2.9. *For $r, \ell \geq 1$, LAC is true for the graph P_{2r}^ℓ.*



(a) $((G \circ \overline{K}_{ns}) - M') \cup M$.



(b) The 1-factor M' in the subgraph induced by $\langle X_5^{ns} \cup X_0^{ns} \rangle$ in $G \circ \overline{K}_{ns}$.



(c) The matching M in $\langle X_5^{ns} \cup X_0^{ns} \rangle$.

(a): Each broken line represents a perfect matching between the respective vertex subsets. Hence the union of all the broken lines represents the matching M in $\langle X_5^{ns} \cup X_0^{ns} \rangle$. Each bold solid line represents a $K_{ns,ns}$ between the respective vertex subsets. Each normal line between X_5^{ns} and X_0^{ns} represents a $K_{ns,ns}$ minus a perfect matching. This graph minus the normal solid lines is isomorphic to the graph obtained from Figure 3 by blowing up each of its vertex by ns independent vertices and replacing each edge by a copy of $K_{ns,ns}$ or a perfect matching of $K_{ns,ns}$ according as the edge is normal solid line or broken line.
(b): Each broken line represents a perfect matching between the respective vertex subsets and the union of all the broken lines represents the perfect matching M' in $\langle X_5^{ns} \cup X_0^{ns} \rangle$.
(c): Each broken line represents a perfect matching between the respective vertex subsets and the union of all the broken lines represents the matching M in $\langle X_5^{ns} \cup X_0^{ns} \rangle$.

FIGURE 6. $((G \circ \overline{K}_{ns}) - M') \cup M$ when $2r = 6$ and $2m = 8$, and it is contained in $(C_6 \times K_8) \circ \overline{K}_{ns}$.

Proof. We prove this lemma in two cases.
Case 1: $\ell = 2k$ for some k.

In P_{2r}^{ℓ}, for each $i, 0 \leq i \leq r-1$, the subgraph $\langle X_{2i} \cup X_{2i+1} \rangle$ of P_{2r}^{ℓ} is isomorphic to $K_{2k,2k}$ and a Hamilton cycle decomposition of $\langle X_{2i} \cup X_{2i+1} \rangle$ is given by

$$\mathscr{H}^i = \{\sigma^0(H_{2i,2i+1}) = H_{2i,2i+1}, \sigma(H_{2i,2i+1}), \sigma^2(H_{2i,2i+1}), \\ \ldots, \sigma^{k-2}(H_{2i,2i+1}), \sigma^{k-1}(H_{2i,2i+1})\},$$

where

$$\begin{aligned}\sigma^0(H_{2i,2i+1}) = H_{2i,2i+1} \\ &= x_{2i,0}\ x_{2i+1,2k-1} x_{2i,1}\ x_{2i+1,2k-2}\ x_{2i,2}\ x_{2i+1,2k-3} \\ &\quad \ldots x_{2i,k-1}\ x_{2i+1,k}\ x_{2i,k}\ x_{2i+1,k-1}\ x_{2i,k+1} x_{2i+1,k-2} \\ &\quad \ldots\ x_{2i,2k-2}\ x_{2i+1,1}\ x_{2i,2k-1}\ x_{2i+1,0}\ x_{2i,0}\end{aligned}$$

and σ denotes the permutation $(x_{2i,0}, x_{2i,1}, x_{2i,2}, x_{2i,3}, \ldots, x_{2i,2k-2}, x_{2i,2k-1})$ $(x_{2i+1,0}, x_{2i+1,1}, x_{2i+1,2}, x_{2i+1,3}, \ldots, x_{2i+1,2k-2}, x_{2i+1,2k-1})$.

First we construct a linear forest L in the graph P_{2r}^{ℓ} as follows and we will generate other linear forests in it using L.

$$\begin{aligned}L = \Big(H_{0,1} - \{x_{0,0}x_{1,0}\}\Big) \cup \Big(H_{2,3} - \{x_{2,0}x_{3,0}\}\Big) \cup \Big(H_{4,5} - \{x_{4,0}x_{5,0}\}\Big) \cup \\ \ldots \cup \Big(H_{2r-2,2r-1} - \{x_{2r-2,0}x_{2r-1,0}\}\Big) \\ \cup \{x_{1,0}x_{2,0}, x_{3,0}x_{4,0},\ x_{5,0}x_{6,0}, \ldots, x_{2r-3,0}x_{2r-2,0}\}\end{aligned}$$

is a Hamilton path of P_{2r}^{ℓ}, see Figure 7(b).

Then $L = \tau^0(L),\ \tau^1(L),\ \tau^2(L),\ \ldots,\ \tau^{k-1}(L)$ gives k Hamilton paths of P_{2r}^{ℓ}, where τ denotes the permutation

$$(x_{0,0}, x_{0,1}, \ldots, x_{0,2k-1})(x_{1,0}, x_{1,1}, x_{1,2}, \ldots, x_{1,2k-1}) \\ (x_{2,0}, x_{2,1}, x_{2,2}, \ldots, x_{2,2k-1})(x_{2r-1,0}, x_{2r-1,1}, x_{2r-1,2}, \ldots, x_{2r-1,2k-1}).$$

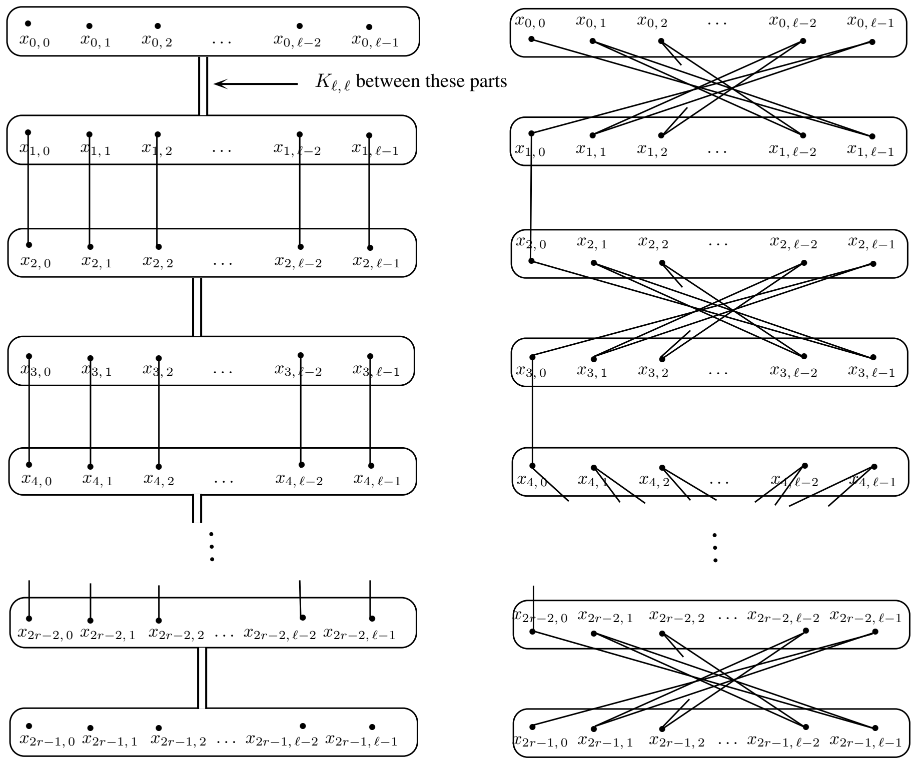
The remaining edges of P_{2r}^{ℓ}, that is, $E(P_{2r}^{\ell}) - (\bigcup_{i=0}^{k-1} \tau^i(L))$, are the edges of the matching

$$R = \bigcup_{i=0}^{r-2} \bigcup_{j=k}^{2k-1} \{x_{2i+1,j} x_{2i+2,j}\} \cup \bigcup_{i=0}^{r-1} \bigcup_{j=0}^{k-1} \{x_{2i,j} x_{2i+1,j}\}.$$

Hence LAC is true for P_{2r}^{ℓ}, if ℓ is even.
Case 2: $\ell = 2k+1$, for some k.

In P_{2r}^{ℓ}, for each $i,\ 0 \leq i \leq r-1$, the subgraph $\langle X_{2i} \cup X_{2i+1} \rangle$ of P_{2r}^{ℓ} is isomorphic to $K_{2k+1,\ 2k+1}$ and a Hamilton cycle decomposition of $\langle X_{2i} \cup X_{2i+1} \rangle$ is given by

$$\mathscr{H}^i = \{\sigma^0(H_{2i,\ 2i+1}) = H_{2i,\ 2i+1},\ \sigma(H_{2i,\ 2i+1}),\ \sigma^2(H_{2i,\ 2i+1}), \\ \ldots, \sigma^{k-2}(H_{2i,\ 2i+1}),\ \sigma^{k-1}(H_{2i,\ 2i+1})\},$$



(a). The graph P_{2r}^{ℓ} (b). The linear forest L in P_{2r}^{ℓ}

Parallel lines between X_{2i} and $X_{2i+1}, 0 \leq i \leq r-1$, in 7$(a)$ represent a $K_{\ell,\ell}$ and normal lines between X_{2i+1} and $X_{2i+2}, 0 \leq i \leq r-2$, denote a perfect matching, which is precisely $\langle X_{2i+1} \cup X_{2i+2} \rangle$.

FIGURE 7. The graphs P_{2r}^{ℓ} and a linear forest L in P_{2r}^{ℓ}.

together with the 1-factor

$$F^{2i,\ 2i+1} = \Big\{x_{2i,0}x_{2i+1,2k-1},\ x_{2i,1}x_{2i+1,2k-2}, x_{2i,2}x_{2i+1,2k-3}, x_{2i,3}x_{2i+1,2k-4}, \\ \ldots, x_{2i,k-1}x_{2i+1,k},\ x_{2i,k}x_{2i+1,k-1},\ x_{2i,k+1}x_{2i+1,k-2}, \\ \ldots, x_{2i,2k-1}x_{2i+1,0},\ x_{2i,2k}x_{2i+1,2k}\Big\},$$

where

$$\sigma^0(H_{2i,2i+1}) = H_{2i,2i+1} = x_{2i,0}\ x_{2i+1,2k}\ x_{2i,1}\ x_{2i+1,2k-1}\ x_{2i,2}\ x_{2i+1,2k-2} \\ \ldots x_{2i,k-1}\ x_{2i+1,k+1}\ x_{2i,k}\ x_{2i+1,k}\ x_{2i,k+1}x_{2i+1,k-1} \\ \ldots\ x_{2i,2k-1}\ x_{2i+1,1}\ x_{2i,2k}\ x_{2i+1,0}\ x_{2i,0}$$

and σ denotes the permutation $(x_{2i,0}, x_{2i,1}, x_{2i,2}, x_{2i,3}, \ldots, x_{2i,2k-2}, x_{2i,2k-1})$ $(x_{2i+1,0}, x_{2i+1,1}, x_{2i+1,2}, x_{2i+1,3}, \ldots, x_{2i+1,2k-2}, x_{2i+1,2k-1})$.

Clearly, $L = \Big(H_{0,1} - \{x_{0,0}x_{1,0}\}\Big) \cup \Big(H_{2,3} - \{x_{2,0}x_{3,0}\}\Big) \cup \Big(H_{4,5} - \{x_{4,0}x_{5,0}\}\Big) \cup \ldots \cup \Big(H_{2r-2,2r-1} - \{x_{2r-2,0}x_{2r-1,0}\}\Big) \cup \{x_{1,0}x_{2,0},\ x_{3,0}x_{4,0},\ x_{5,0}x_{6,0}, \ldots, x_{2r-3,0}x_{2r-2,0}\}$ is a Hamilton path of the graph P_{2r}^{ℓ}.

Now $\mathscr{L} = \Big\{L = \rho^0(L),\ \rho^1(L),\ \rho^2(L),\ \ldots,\ \rho^{k-1}(L)\Big\}$ is the set of k edge disjoint linear forests of P_{2r}^{ℓ}, where ρ denotes the permutation

$$(x_{0,0}, x_{0,1}, x_{0,2}, \ldots, x_{0,2k-1}, x_{0,2k})(x_{1,0}, x_{1,1}, x_{1,2}, \ldots, x_{1,2k-1}, x_{1,2k})$$
$$(x_{2,0}, x_{2,1}, x_{2,2}, \ldots, x_{2,2k-1}, x_{2,2k})$$
$$\ldots(x_{2r-1,0}, x_{2r-1,1}, x_{2r-1,2}, \ldots, x_{2r-1,2k-1}, x_{2r-1,2k}).$$

The edges not contained in the subgraphs of $\mathscr{L}$ are $R = \bigcup_{i=0}^{r-1}\{F^{2i,2i+1}\} \cup \bigcup_{i=0}^{r-1}\bigcup_{j=0}^{k-1}\{x_{2i,j}x_{2i+1,j}\} \cup \bigcup_{i=0}^{r-2}\bigcup_{j=k}^{2k}\{x_{2i+1,j}x_{2i+2,j}\}$. Clearly, $\mathscr{L} \cup R$ is a linear forest decomposition of P_{2r}^{ℓ} and hence LAC is true for P_{2r}^{ℓ}.

This completes the proof of the lemma. □

Next we define the graphs $Z_{2r,2m}$ and $Z_{2r,2m}^{ns}$ and show that LAC is true for these graphs.

The graph $Z_{2r,2m}$ is obtained from the path $P_{2r} = x_0x_1 \ldots x_{2r-1}$ as follows: replace each vertex x_i of P_{2r} by the set of $2m$ independent vertices $X_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,2m-1}\}$. The edge set of $Z_{2r,2m}$ consists of the following edges: for $0 \le i \le r-1, \langle X_{2i} \cup X_{2i+1}\rangle$ induces a $K_{2m,2m} - F_0(X_{2i}, X_{2i+1})$ and for $0 \le i \le r-2, \langle X_{2i+1} \cup X_{2i+2}\rangle$ induces the 1-factor $F_{2m-1}(X_{2i+1}, X_{2i+2})$. The graph $Z_{2r,2m}^{ns}$ is obtained from $Z_{2r,2m}$ by replacing each of its vertex $x_{a,b}, 0 \le a \le 2r-1, 0 \le b \le 2m-1$, by an independent set $X_{a,b}$ of ns vertices. If $x_{a,b}x_{c,d}$ is an edge of $\langle X_{2i} \cup X_{2i+1}\rangle \subset Z_{2r,2m}$, then $\langle X_{a,b} \cup X_{c,d}\rangle$ is isomorphic to $K_{ns,ns}$ and if $x_{a,b}x_{c,d}$ is an edge in $\langle X_{2i+1} \cup X_{2i+2}\rangle$, then $\langle X_{a,b} \cup X_{c,d}\rangle$ is isomorphic to a 1-factor of $K_{ns,ns}$ with partite sets $X_{a,b}$ and $X_{c,d}$, that is, we can consider $Z_{2r,2m}^{ns}$ as a "blown up" graph of $Z_{2r,2m}$ obtained by blowing up each edge e of $Z_{2r,2m}$ into a copy of $K_{ns,ns}$ or a 1-factor of $K_{ns,ns}$, according as the edge e belongs to $\langle X_{2i} \cup X_{2i+1}\rangle$, or $\langle X_{2i+1} \cup X_{2i+2}\rangle$, respectively. For the subgraph Z of $Z_{2r,2m}$, let $B(Z)$ denote the subgraph of $Z_{2r,2m}^{ns}$ obtained by blowing up each vertex of Z by ns independent vertices and the edge $e \in Z$ by a copy of $K_{ns,ns}$ or a 1-factor of $K_{ns,ns}$, according as the edge $e \in \langle X_{2i} \cup X_{2i+1}\rangle$ or $e \in \langle X_{2i+1} \cup X_{2i+2}\rangle$, respectively.

It is known that LAC is true for $Z_{2r,2m}$ by [13]. But we need an alternate proof for our future reference, which is contained in the proof of the following lemma.

Lemma 2.10. *For $r, s \ge 2$ and $ns \ne 2, 6$, LAC is true for the graph $Z_{2r,2m}^{ns}$.*

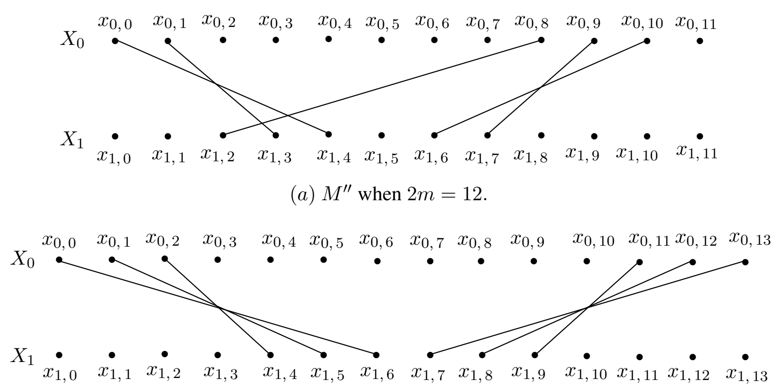
Proof. Consider the graph $Z_{2r,2m}$. In $Z_{2r,2m}$, for $0 \le i \le r-1$, the subgraph $\langle X_{2i} \cup X_{2i+1}\rangle$ is isomorphic to $K_{2m,2m} - F_0(X_{2i}, X_{2i+1})$, where F_0 is the 1-factor of jump zero and, for $0 \le i \le r-2$, the subgraph $\langle X_{2i+1} \cup X_{2i+2}\rangle$ is isomorphic to $F_{2m-1}(X_{2i+1}, X_{2i+2})$. Consider the subgraph $\langle X_0 \cup X_1\rangle \cong$

$K_{2m,2m} - F_0$ of $Z_{2r,2m}$, where $X_0 = \{x_{0,0}, x_{0,1}, x_{0,2}, \ldots, x_{0,2m-1}\}$ and $X_1 = \{x_{1,0}, x_{1,1}, x_{1,2}, \ldots, x_{1,2m-1}\}$. Let

$$M'' = \begin{cases} \{x_{0,0}x_{1,m-1}, x_{0,1}x_{1,m-2}, x_{0,2}x_{1,m-3}, \ldots, x_{0,\frac{m-3}{2}}x_{1,\frac{m+1}{2}}\} \\ \cup\{x_{0,2m-1}x_{1,m}, x_{0,2m-2}x_{1,m+1}, x_{0,2m-3}x_{1,m+2}, \\ \ldots, x_{0,\frac{3m+1}{2}}x_{1,\frac{3m-3}{2}}\}, \qquad \text{if } 2m \equiv 2 \pmod 4 \\ \{x_{0,0}x_{1,m-2}, x_{0,1}x_{1,m-3}, x_{0,2}x_{1,m-4}, \ldots, x_{0,\frac{m}{2}-2}x_{1,\frac{m}{2}}\} \\ \cup\{x_{0,2m-2}x_{1,m}, x_{0,2m-3}x_{1,m+1}, x_{0,2m-4}x_{1,m+2}, \\ \ldots, x_{0,\frac{3m}{2}}x_{1,\frac{3m-4}{2}}\} \cup \{x_{0,\frac{3m-2}{2}}x_{1,\frac{m}{2}-1}\}, \qquad \text{if } 2m \equiv 0 \pmod 4\,. \end{cases}$$

It can be observed that M'' contains $m-1$ edges of distinct even jumps, namely, $2, 4, 6 \ldots, 2m-2$ in $\langle X_0 \cup X_1 \rangle$, see for example, Figure 8.

Now we explain the idea behind the proof of this lemma. First we shall prove that $Z_{2r,2m}$ has $m-1$ edge-disjoint Hamilton paths and a linear forest, disjoint from the Hamilton paths. That is, $Z_{2r,2m}$ has m linear forests $\{L_1, L_2, \ldots, L_m\}$. Then we blow up each $L_j, 1 \le j \le m-1$, that is, we find $B(L_j)$ and, if necessary, we add some edges of $B(L_m)$ to $B(L_j)$ and then we decompose the resulting graph into ns Hamilton paths of $Z^{ns}_{2r,2m}$. Finally, we prove that the set of edges not in the above set of $(m-1)ns$ Hamilton paths of $Z^{ns}_{2r,2m}$ has a decomposition into $\lfloor \frac{ns+1}{2} \rfloor$ linear forests.



(b) M'' when $2m = 14$.

FIGURE 8. The matching M'' when $2m \equiv 0 \pmod 4$ and $2m \equiv 2 \pmod 4$ for $2m = 12$ and $2m = 14$.

Now we shall give a detailed proof of the above sketch.

Claim. *$Z_{2r,2m}$ admits a linear forest decomposition into $m-1$ Hamilton paths and a linear forest.*

Proof of claim. In $Z_{2r,2m}$, let M'' be the matching in $\langle X_0 \cup X_1 \rangle$ described above. Clearly $M''(= \rho^0(M''))$, $\rho^{m-1}(M'')$, $\rho^{2(m-1)}(M'')$, $\ldots$, $\rho^{(r-1)(m-1)}(M'')$ are matchings in $\langle X_0 \cup X_1 \rangle$, where ρ denotes the permutation

$(x_{0,0}, x_{0,1}, \ldots, x_{0,2m-1})(x_{1,0}, x_{1,1}, \ldots, x_{1,2m-1})$ on the vertices $X_0 \cup X_1$. Then by abuse of notation we denote by $M''(= \rho_0^0(M''))$, $\rho_1^{m-1}(M'')$, $\rho_2^{2(m-1)}(M'')$, $\rho_3^{3(m-1)}(M'')$, $\ldots$, $\rho_{r-1}^{(r-1)(m-1)}(M'')$ the matchings in $\langle X_0 \cup X_1\rangle$, $\langle X_2 \cup X_3\rangle$, $\langle X_4 \cup X_5\rangle$, $\ldots$ $\langle X_{2r-2} \cup X_{2r-1}\rangle$, respectively, that is, $\rho_k^{k(m-1)}(M'')$ is a matching of $\langle X_{2k} \cup X_{2k+1}\rangle$ whose edges are the corresponding edges of $\rho^{k(m-1)}(M'') \subset \langle X_0 \cup X_1\rangle$; that is, if the edge $x_{0,i}x_{1,i+\ell}$ is in $\langle X_0 \cup X_1\rangle$, then $x_{2k,i}x_{2k+1,i+\ell}$ is called its corresponding edge in $\langle X_{2k} \cup X_{2k+1}\rangle$.

Now we delete the edges of the matchings M'', $\rho_1^{m-1}(M'')$, $\rho_2^{2(m-1)}(M'')$, $\rho_3^{3(m-1)}(M'')$, $\ldots$, $\rho_{r-1}^{(r-1)(m-1)}(M'')$ in $\langle X_0 \cup X_1\rangle$, $\langle X_2 \cup X_3\rangle$, $\langle X_4 \cup X_5\rangle$, $\ldots$, $\langle X_{2r-2} \cup X_{2r-1}\rangle$, respectively. As $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$, is isomorphic to $K_{2m,2m} - F_0$, it has a Hamilton cycle decomposition $\{H^j_{2i,2i+1} | 1 \le j \le m-1\} \cup F_1(X_{2i}, X_{2i+1})$, where $F_1(X_{2i}, X_{2i+1})$ is a 1-factor of jump 1 in $\langle X_{2i} \cup X_{2i+1}\rangle$, and $H^j_{2i,2i+1} = F_{2j}(X_{2i}, X_{2i+1}) \cup F_{2j+1}(X_{2i}, X_{2i+1}), 1 \le j \le m-1$. Clearly, by the choice of M'', M'' is orthogonal to the Hamilton cycles $\{H^j_{0,1} | 1 \le j \le m-1\}$, since M'' contains exactly one edge of jump $2, 4, 6, \ldots, 2m-2$. As M'' is orthogonal to the Hamilton cycles in $\langle X_0 \cup X_1\rangle$, the matching $\rho_k^{k(m-1)}(M'')$ is orthogonal to the Hamilton cycles $\{H^j_{2k,2k+1} | 1 \le j \le m-1\}$ in $\langle X_{2k} \cup X_{2k+1}\rangle, 0 \le k \le r-1$.

Let $D_j, 1 \le j \le m-1$, denote the set of r edges, one edge from each of the matchings $\rho_i^{i(m-1)}(M'') \subset \langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$, such that its edges are of jumps $2j$ and $2m-2j$, alternately, from $\rho_i^{i(m-1)}(M'') \subset \langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$, that is,

$$D_j = \begin{cases} \left(H^j_{0,1} \cap M''\right) \cup \left(H^{m-j}_{2,3} \cap \rho_1^{m-1}(M'')\right) \cup \left(H^j_{4,5} \cap \rho_2^{2(m-1)}(M'')\right) \\ \cup \left(H^{m-j}_{6,7} \cap \rho_3^{3(m-1)}(M'')\right) \cup \ldots \\ \cup \left(H^{m-j}_{2r-4,2r-3} \cap \rho_{r-2}^{(r-2)(m-1)}(M'')\right) \\ \cup \left(H^{j}_{2r-2,2r-1} \cap \rho_{r-1}^{(r-1)(m-1)}(M'')\right), \qquad \text{if } 2r \equiv 2 \pmod 4, \\ \left(H^j_{0,1} \cap M''\right) \cup \left(H^{m-j}_{2,3} \cap \rho_1^{m-1}(M'')\right) \cup \left(H^j_{4,5} \cap \rho_2^{2(m-1)}(M'')\right) \\ \cup \left(H^{m-j}_{6,7} \cap \rho_3^{3(m-1)}(M'')\right) \cup \ldots \\ \cup \left(H^{j}_{2r-4,2r-3} \cap \rho_{r-2}^{(r-2)(m-1)}(M'')\right) \\ \cup \left(H^{m-j}_{2r-2,2r-1} \cap \rho_{r-1}^{(r-1)(m-1)}(M'')\right), \qquad \text{if } 2r \equiv 0 \pmod 4, \end{cases}$$

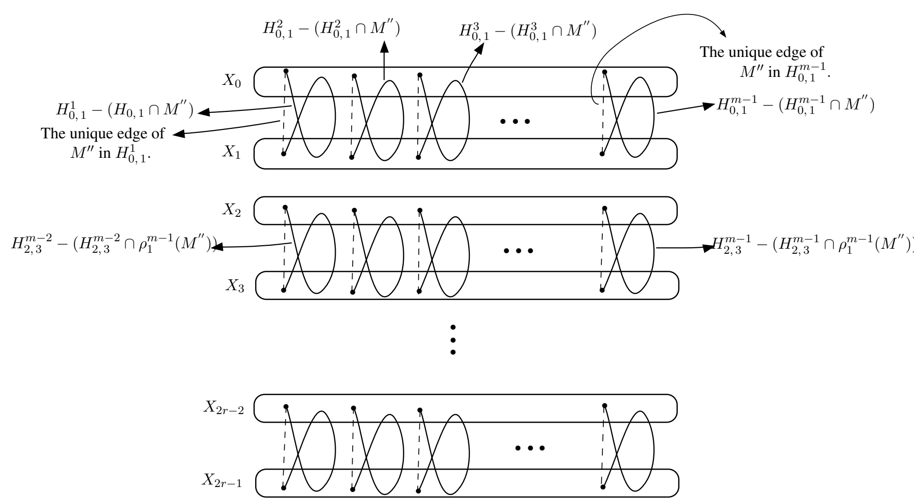
see Figure 9.

Next, we construct a linear forest $L_j, 1 \le j \le m-1$, in $Z_{2r,2m}$ as follows: before that we construct the subgraph L'_j of $Z_{2r,2m}$.

Let L'_j denote the union of r Hamilton paths, one from each of the subgraphs $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$, such that it contains the jth and $(m-j)$th Hamilton paths, alternately, from $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$, see Figure 9, that is

$$L'_j = \begin{cases} \left(H^j_{0,1} \cup H^{m-j}_{2,3} \cup H^j_{4,5} \cup H^{m-j}_{6,7} \cup \ldots \cup H^{m-j}_{2r-4,2r-3} \cup H^j_{2r-2,2r-1}\right) \\ \qquad -D_j, \text{if } 2r \equiv 2 \pmod 4) \\ \left(H^j_{0,1} \cup H^{m-j}_{2,3} \cup H^j_{4,5} \cup H^{m-j}_{6,7} \cup \ldots \oplus H^j_{2r-4,2r-3} \cup H^{m-j}_{2r-2,2r-1}\right) \\ \qquad -D_j, \text{if } 2r \equiv 0 \pmod 4), \end{cases}$$

in other words, we have punched one edge of the orthogonal matching $\rho_i^{i(m-1)}(M'')$ from the specified Hamilton cycle of $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$.



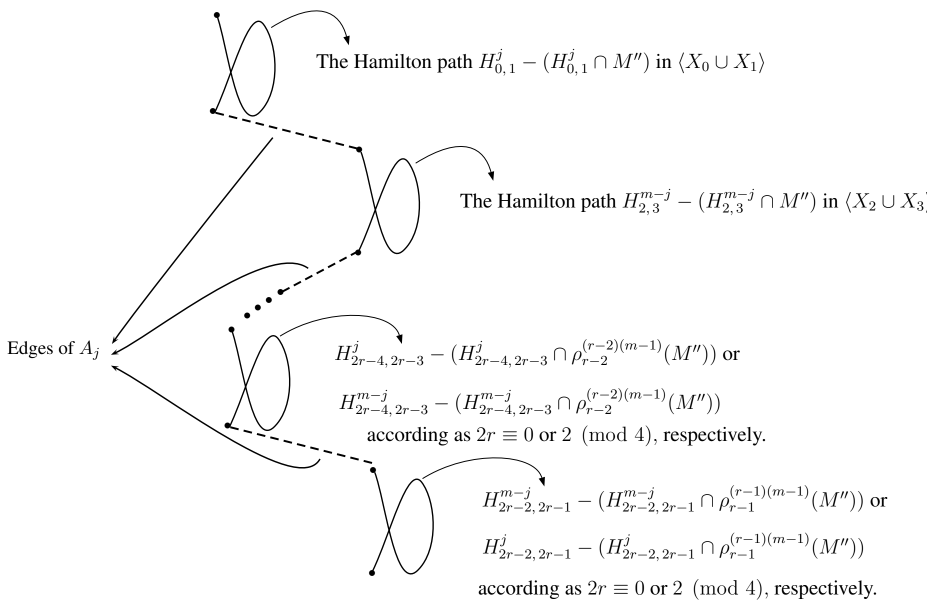
For clarity, the Hamilton cycles in the Hamilton cycle decomposition of $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$, are shown "like loops" together with a broken line between X_{2i} and X_{2i+1}; (the ends of the broken edges are not necessarily the "corresponding vertices"). Each D_j contains exactly one broken line of $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$. $\bigcup_{j=1}^{m-1} D_j$ denotes all the broken lines. Also $\bigcup_{j=1}^{m-1} L'_j$ denotes the disjoint union of all the paths obtained by deleting the broken lines from the Hamilton cycles shown in $\langle X_{2i} \cup X_{2i+1}\rangle, 0 \le i \le r-1$.

FIGURE 9.

Next we shall show that the edges in $\bigcup_{i=0}^{r-2} \langle X_{2i+1} \cup X_{2i+2}\rangle$ can be partitioned into m sets $A_j, 1 \le j \le m$ so that each $L'_j \cup A_j, 1 \le j \le m-1$, is a Hamilton path of $Z_{2r,2m}$ and $\bigcup_{i=0}^{r-1} F_1(X_{2i}, X_{2i+1}) \cup \bigcup_{j=1}^{m-1} D_j \cup A_m$ is a linear forest (note that $\bigcup_{j=1}^{m-1} D_j$ is a matching of $\bigcup_{i=0}^{r-1} \langle X_{2i} \cup X_{2i+1}\rangle$).

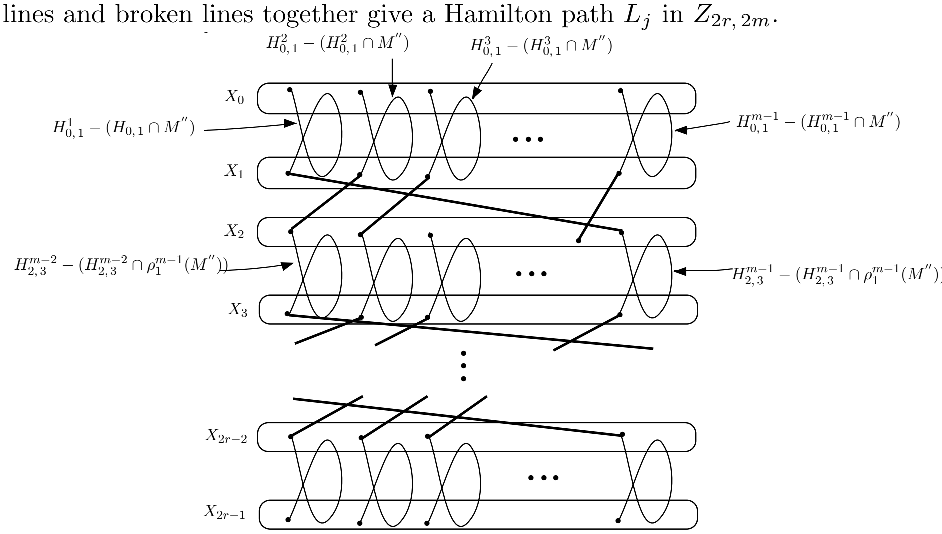
Consider $H^j_{0,1}$ and M'' in $\langle X_0 \cup X_1\rangle$ and let $x_{0,a}x_{1,a+2j}$ be an edge of jump $2j$ in $\langle X_0 \cup X_1\rangle$; by the choice of M'', the edge of jump $2m-2j$ of M'' is $x_{0,a+2j+m}x_{1,a+m}$. Now

$$\begin{aligned} \rho_1^{m-1}(x_{2,a+2j+m}x_{3,a+m}) &= x_{2,a+2j+m+m-1}x_{3,a+m+m-1} \\ &= x_{2,a+2j-1}x_{3,a-1}. \end{aligned}$$



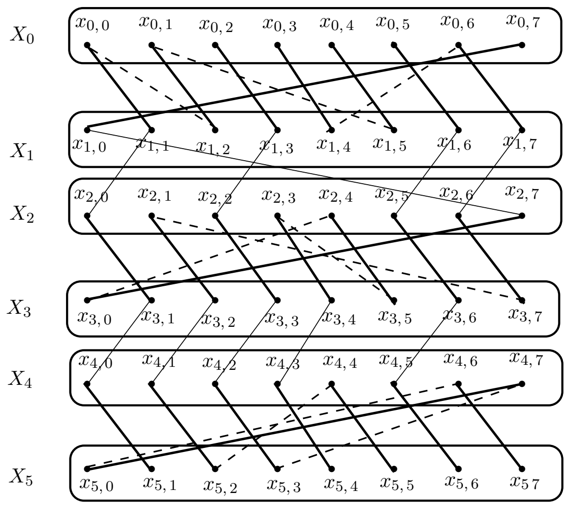
$10(a)$: The linear forest $L_j = L'_j \cup A_j$.

Each "loop like" solid line represents a Hamilton path in $\langle X_{2i} \cup X_{2i+1} \rangle$, $0 \le i \le r-1$. Each broken line represents an edge of A_j in $\langle X_{2i+1} \cup X_{2i+2} \rangle$. The "loop like" solid lines and broken lines together give a Hamilton path L_j in $Z_{2r,2m}$.



$10(b)$

FIGURE 10. Various Hamilton paths of $Z_{2r,2m}$ are shown, wherein a Hamilton path of $\langle X_{2i} \cup X_{2i+1} \rangle$ is shown "like a loop" (between X_{2i} and X_{2i+1}) for clarity. The bold edges between X_{2i+1} and $X_{2i+2}, 0 \le i \le r-2$, denote $\bigcup_{j=1}^{m-1} A_j$.



Broken lines in $\langle X_0 \cup X_1 \rangle$ denote M''. In $\langle X_{2i} \cup X_{2i+1} \rangle, i = 1, 2$, the broken lines denote the edges of $\rho_1^{(m-1)}(M'')$ and $\rho_2^{2(m-1)}(M'')$, respectively. The solid (bold and normal) lines are the edges that are not contained in the linear forests $L_j, 1 \leq j \leq 3$ of $Z_{6,8}$.

FIGURE 11. The linear forest L_4 of $Z_{6,8}$.

Hence $H_{0,1}^j - (H_{0,1}^j \cap M'')$ and $H_{2,3}^{m-j} - (H_{2,3}^{m-j} \cap \rho_1^{(m-1)}(M''))$, that is, deleting the unique edge of M'' in $H_{0,1}^j$ and deleting the unique edge of $\rho_1^{(m-1)}(M'')$ in $H_{2,3}^{m-j}$ we get Hamilton paths in $\langle X_0 \cup X_1 \rangle$ and $\langle X_2 \cup X_3 \rangle$, respectively, and the edge $x_{1,a+2j} x_{2,a+2j-1}$ of $\langle X_1 \cup X_2 \rangle$ joins the end vertex of $H_{0,1}^j - (H_{0,1}^j \cap M'')$ and the origin of $H_{2,3}^{m-j} - (H_{2,3}^{m-j} \cap \rho_1^{m-1}(M''))$. In general, $H_{2k,2k+1}^j - (H_{2k,2k+1}^j \cap \rho_k^{k(m-1)}(M''))$ and $H_{2k+2,2k+3}^{m-j} - (H_{2k+2,2k+3}^{m-j} \cap \rho_{k+1}^{(k+1)(m-1)}(M''))$ are Hamilton paths in $\langle X_{2k} \cup X_{2k+1} \rangle$ and $\langle X_{2k+2} \cup X_{2k+3} \rangle$, respectively, and there is an edge of $\langle X_{2k+1} \cup X_{2k+2} \rangle$ joining the end of $H_{2k,2k+1}^j - (H_{2k,2k+1}^j \cap \rho_k^{k(m-1)}(M''))$ and the origin of $H_{2k+2,2k+3}^{m-j} - (H_{2k+2,2k+3}^{m-j} \cap \rho_{k+1}^{(k+1)(m-1)}(M''))$, see Figures 10(a) and 10(b). The above is true for all $k, 0 \leq k \leq r-2$. Let the union of the edges in $\langle X_{2i+1} \cup X_{2i+2} \rangle$ connecting the paths in L'_j be denoted by A_j, see Figure 10(a). Hence $L'_j \cup A_j$ is a Hamilton path of $Z_{2r,2m}$, see Figure 10(a). Let $Z = \bigcup_{i=0}^{r-2} \left(\langle X_{2i+1} \cup X_{2i+2} \rangle \right) - \bigcup_{j=1}^{m-1} A_j$, that is the edges of $\bigcup_{i=0}^{r-2} \langle X_{2i+1} \cup X_{2i+2} \rangle$ which are not on the Hamilton paths of $Z_{2r,2m}$ obtained above.

Thus we have $(m-1)$ linear forests $L_j = L'_j \cup A_j, 1 \leq j \leq m-1$, where each L_j is a Hamilton path of $Z_{2r,2m}$, see Figure 10(b).

By the way M'' is constructed in $\langle X_0 \cup X_1 \rangle$, the edges not on $\bigcup_{i=1}^{m-1} L_i$ in $Z_{2r,2m}$ induce a linear forest, say L_m, which consists of vertex disjoint union of paths, see Figure 11, for $2r = 6, 2m = 8$. Thus we have decomposed

$Z_{2r,2m}$ into $m-1$ Hamilton paths and a linear forest. This completes the proof of the claim. □

Now using the linear forest decomposition of $Z_{2r,2m}$, we shall decompose $Z^{ns}_{2r,2m}$ into linear forests by blowing up each of the graphs $L_j \cup D_j, 1 \leq j \leq m-1$, of $Z_{2r,2m}$, see Figure 12, into ns linear forests and a matching. Further, we shall show that the edges not contained in the $(m-1)ns$ linear forests of $Z^{ns}_{2r,2m}$ induce $\lceil \frac{ns+1}{2} \rceil$ linear forests.

First we consider the subgraph $L'_j \cup D_j$ of $Z_{2r,2m}$, that is,

$$L'_j \cup D_j = \begin{cases} H^j_{0,1} \cup H^{m-j}_{2,3} \cup H^j_{4,5} \cup H^{m-j}_{6,7} \cup \ldots \cup H^{m-j}_{2r-4,2r-3} \\ \qquad \cup H^j_{2r-2,2r-1}, \text{if } 2r \equiv 2 \pmod 4 \\ H^j_{0,1} \cup H^{m-j}_{2,3} \cup H^j_{4,5} \cup H^{m-j}_{6,7} \cup \ldots \cup H^j_{2r-4,2r-3} \\ \qquad \cup H^{m-j}_{2r-2,2r-1}, \text{if } 2r \equiv 0 \pmod 4, \end{cases}$$

The graph $H^j_{2k,2k+1} \circ \overline{K}_{ns}, 0 \leq k \leq r-1, 1 \leq j \leq m-1$, subgraph of $Z^{ns}_{2r,2m}$, has a Hamilton cycle decomposition such that a matching $M^j_{2k,2k+1}$ in $\langle X^{ns}_{2k} \cup X^{ns}_{2k+1} \rangle$ is orthogonal to the Hamilton cycle decomposition of $H^j_{2k,2k+1} \circ \overline{K}_{ns}$; we can suppose that $M^j_{2k,2k+1}$ is contained in the complete bipartite subgraph of $H^j_{2k,2k+1} \circ \overline{K}_{ns}$ induced by the layers corresponding to the ends of the unique edge of $H^j_{2k,2k+1} \cap \rho_k^{k(m-1)}(M'')$, by Lemma 2.7.

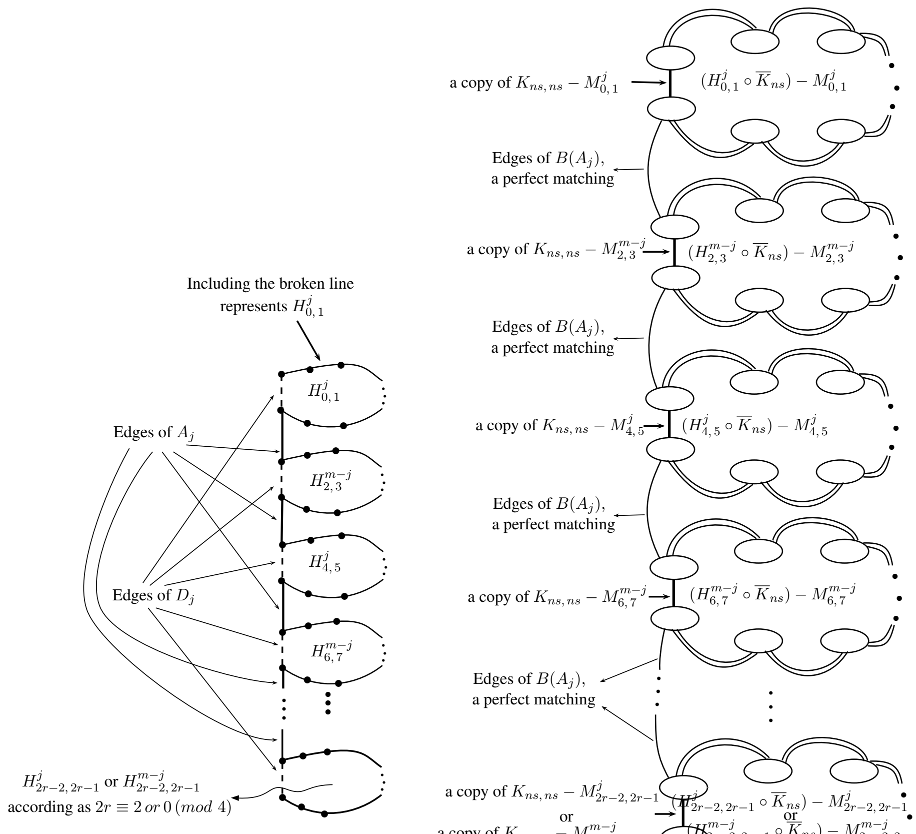
Thus, as $L'_j \cup D_j$ is the disjoint union of cycles, each component $(L'_j \cup D_j) \circ \overline{K}_{ns}$ has a Hamilton path decomposition together with a matching. Let $L'_{j,ns}$ denote the union of these paths, that is,

$$L'_{j,ns} = \begin{cases} ((L'_j \cup D_j) \circ \overline{K}_{ns}) - (M^j_{0,1} \cup M^{m-j}_{2,3} \cup M^j_{4,5} \cup M^{m-j}_{6,7} \\ \qquad \cup \ldots \cup M^{m-j}_{2r-4,2r-3} \cup M^j_{2r-2,2r-1}), \text{if } 2r \equiv 2 \pmod 4 \\ ((L'_j \cup D_j) \circ \overline{K}_{ns}) - \Big(M^j_{0,1} \cup M^{m-j}_{2,3} \cup M^j_{4,5} \cup M^{m-j}_{6,7} \\ \qquad \cup \ldots \cup M^j_{2r-4,2r-3} \cup M^{m-j}_{2r-2,2r-1}\Big), \text{if } 2r \equiv 0 \pmod 4. \end{cases}$$

As each component of $L'_{j,ns}$ has a decomposition into Hamilton paths when they get fused with the edges of $B(A_j)$, where $B(A_j)$ is defined just above this lemma, we get ns edge disjoint Hamilton paths of $Z^{ns}_{2r,2m}$, that is, the subgraph $L'_{j,ns} \cup B(A_j)$ is exactly the union of ns edge disjoint Hamilton paths in $Z^{ns}_{2r,2m}$, see Figure 12. As j varies from 1 to $m-1$, we have $(m-1)ns$ edge disjoint Hamilton paths in $Z^{ns}_{2r,2m}$. Let $\mathscr{L}$ denote these $(m-1)ns$ Hamilton paths of $Z^{ns}_{2r,2m}$.

The set of edges of $Z^{ns}_{2r,2m}$ not contained in the $(m-1)ns$ Hamilton paths of $Z^{ns}_{2r,2m}$, obtained above, are

$$\left(\bigcup_{j=0}^{r-1} (F_1(X_{2j}, X_{2j+1}) \circ \overline{K}_{ns})\right) \cup \left(\bigcup_{j=1}^{m-1} \bigcup_{i=0}^{r-1} M^j_{2i,2i+1}\right) \cup B(Z),$$



$12(a)$ The graph $L_j \cup D_j$

$12(b)$ The graph $L'_{j,ns} \cup B(A_j)$

$12(a)$. The graph $L_j \cup D_j$. In this figure, each normal line represents a Hamilton path in $\langle X_{2k} \cup X_{2k+1} \rangle, 0 \le k \le r-1$, the bold lines represent the edges of A_j and the broken lines represent the edges of D_j.
$12(b)$. The graph $L'_{j,ns} \cup B(A_j)$, where each small round represents an independent set of ns vertices. Each one of the parallel lines represents a complete bipartite subgraph $K_{ns,ns}$ between the respective vertex subsets. Also each of the bold lines denotes a copy of $K_{ns,ns}$ minus a perfect matching $M^j_{2i,2i+1}$ or $M^{m-j}_{2i,2i+1}$, according as i is even or odd, respectively, in it; this matching $M^j_{2i,2i+1}$ or $M^{m-j}_{2i,2i+1}$ is orthogonal to the Hamilton cycles of the Hamilton cycle decomposition of $H^j_{2i,2i+1} \circ \overline{K}_{ns}$ or $H^{m-j}_{2i,2i+1} \circ \overline{K}_{ns}$, respectively. The bold lines together with the parallel lines induce ns Hamilton paths in respective subgraphs $\langle X^{ns}_{2k} \cup X^{ns}_{2k+1} \rangle, 0 \le k \le r-1$, of $Z^{ns}_{2r,2m}$ and their union is $L'_{j,ns}$. Each of the normal lines represents the edges of $B(A_j)$, a perfect matching between the respective parts.

FIGURE 12. The graph $L_j \cup D_j$ of $Z_{2r,2m}$ and the corresponding Hamilton paths in $Z^{ns}_{2r,2m}$.

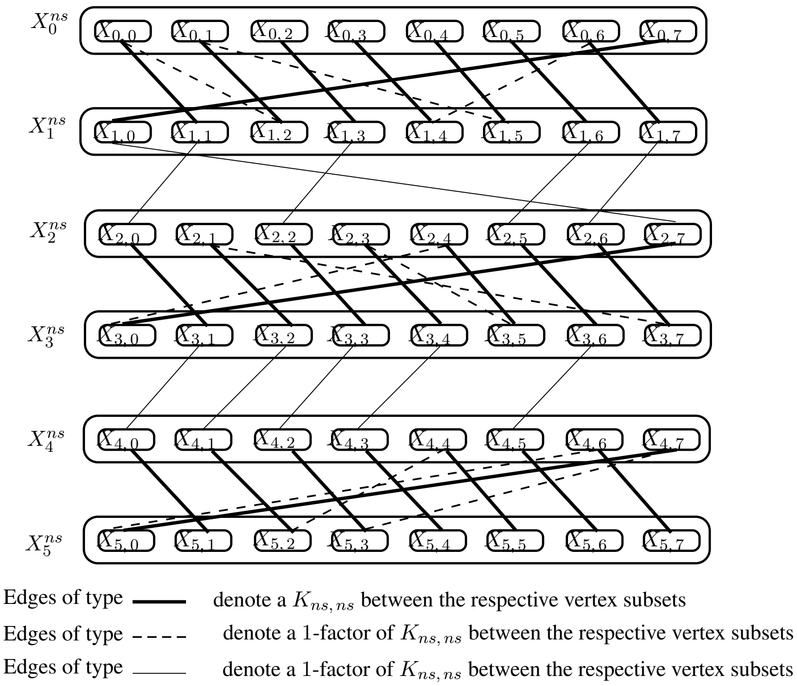


FIGURE 13. The graph $\left(\bigcup_{j=0}^{r-1}(F_1(X_{2j}, X_{2j+1}) \circ \overline{K}_{ns})\right) \cup \left(\bigcup_{j=1}^{m-1}\bigcup_{i=0}^{r-1} M_{2i,2i+1}^{j}\right) \cup B(Z)$ when $2r = 6$ and $2m = 8$.

where $Z = \bigcup_{i=0}^{r-2}\left(\langle X_{2i+1} \cup X_{2i+2}\rangle\right) - \bigcup_{j=1}^{m-1} A_j$ of $Z_{2r,2m}$; thus the leftover edges look like the graph in Figure 13 (Figure 13 is obtained from the graph of Figure 11 by blowing up each one of the bold solid lines into a $K_{ns,ns}$ and blowing up each one of the normal solid lines and each broken lines in it by a perfect matching between the respective vertex subsets). Note that $\bigcup_{j=1}^{m-1} M_{2k,2k+1}^{j}$ is the union of $m-1$ matchings which corresponds to the edges of $\rho_k^{k(m-1)}(M'')$ in $\langle X_{2k} \cup X_{2k+1}\rangle$.

Consider the graph

$$\left(\bigcup_{j=0}^{r-1}(F_1(X_{2j}, X_{2j+1}) \circ \overline{K}_{ns})\right) \cup \left(\bigcup_{j=1}^{m-1}\bigcup_{i=0}^{r-1} M_{2i,2i+1}^{j}\right) \cup B(Z).$$

Recall that if $x_{a,b} \in Z_{2r,2m}$, then $X_{a,b}$ denotes the set of vertices obtained by replacing $x_{a,b}$ by ns independent vertices. Clearly, $\langle X_{a,b} \cup X_{c,d}\rangle$ is (i) a complete bipartite graph $K_{ns,ns}$, if $x_{a,b}x_{c,d}$ is an edge of $F_1(X_{2j}, X_{2j+1}) \subset Z_{2r,2m}, 0 \le j \le r-1$, or (ii) a 1-factor between $X_{a,b}$ and $X_{c,d}$ in $Z_{2r,2m}^{ns}$, if $x_{a,b}x_{c,d} \in E(Z_{2r,2m})$ is the corresponding edge of at least one matching in $\bigcup_{j=1}^{m-1}\bigcup_{i=0}^{r-1} M_{2i,2i+1}^{j}$, or (iii) graph without edges, if $x_{a,b}x_{c,d}$ is not an edge in

$$\left(\bigcup_{j=0}^{r-1}(F_1(X_{2j},X_{2j+1})\circ\overline{K}_{ns})\right)\cup\left(\bigcup_{j=1}^{m-1}\bigcup_{i=0}^{r-1}M_{2i,2i+1}^{j}\right)\cup B(Z).$$

Hence if $\langle X_{a,b}\cup X_{c,d}\rangle$ is a 1-factor then, relabel $X_{a,b}$ or $X_{c,d}$ so that the 1-factor is of jump 0 between them. Now $\bigcup_{j=0}^{r-1}(F_1(X_{2j},X_{2j+1})\circ\overline{K}_{ns})\oplus\bigcup_{j=1}^{m-1}\bigcup_{i=0}^{r-1}M_{2i,2i+1}^{j}\oplus B(Z)$ is the union of vertex disjoint copies of P_{2q}^{ns}, where P_{2q}^{ns} is as defined just before Lemma 2.9, for different values of q. But each copy of the form P_{2q}^{ns} has a decomposition into $\lceil\frac{ns+1}{2}\rceil$ linear forests, by Lemma 2.9.

Thus totally we have $(m-1)ns+\lceil\frac{ns+1}{2}\rceil=\left\lceil\frac{\Delta(Z_{2r,2m}^{ns})+1}{2}\right\rceil$ linear forests in $Z_{2r,2m}^{ns}$ and hence LAC is true for $Z_{2r,2m}^{ns}$.

This completes the proof of the lemma. ☐

Lemma 2.11. *For $r,m\geq 2$, and $ns\neq 2,6$, LAC is true for the graph $(K_{2r}\circ\overline{K}_s)\times(K_{2m}\circ\overline{K}_n)$.*

Proof. If $n=s=1$, then the proof follows from Lemma 2.2 and hence we assume that $n,s>1$. Observe that the graph $(K_{2r}\circ\overline{K}_s)\times(K_{2m}\circ\overline{K}_n)\cong(K_{2r}\times K_{2m})\circ\overline{K}_{ns}$. The graph K_{2r} has a decomposition into $r-1$ Hamilton cycles $\mathscr{C}=\{C_{2r}^1,C_{2r}^2,\ldots,C_{2r}^{r-1}\}$ and a perfect matching $F^{2r}=\{x_0x_1,x_2x_3,x_4x_5,\ldots,x_{2r-2}x_{2r-1}\}$, where $V(K_{2r})=\{x_0,x_1,\ldots,x_{2r-2},x_{2r-1}\}$. Further, there exists a matching M^{2r} of K_{2r} such that M^{2r} is orthogonal to the Hamilton cycles in $\mathscr{C}$ and $M^{2r}\cup F^{2r}$ is a Hamilton path of K_{2r}, by Lemma 2.2 of [13].

Thus

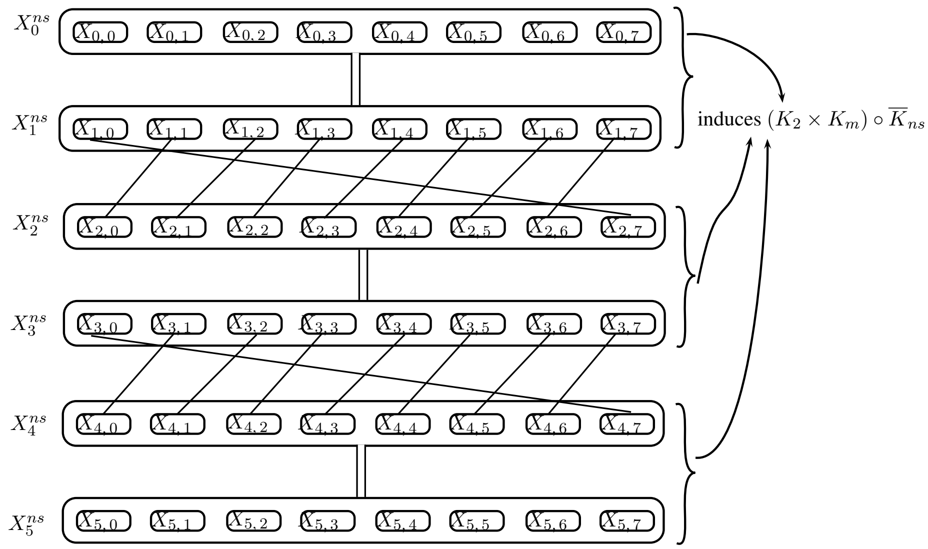
(2.1)

$$\begin{aligned}(K_{2r}\times K_{2m})\circ\overline{K}_{ns}&=((C_{2r}^1\oplus C_{2r}^2\oplus\ldots\oplus C_{2r}^{r-1}\oplus F^{2r})\times K_{2m})\circ\overline{K}_{ns}\\&=\Big(\oplus_{i=1}^{r-1}(C_{2r}^i\times K_{2m})\circ\overline{K}_{ns}\Big)\oplus\Big((F^{2r}\times K_{2m})\circ\overline{K}_{ns}\Big)\\&=G_1\oplus G_2\end{aligned}$$

where $G_1=(\oplus_{i=1}^{r-1}(C_{2r}^i\times K_{2m})\circ\overline{K}_{ns}\Big)$ and $G_2=\Big((F^{2r}\times K_{2m})\circ\overline{K}_{ns})$.

First, we explain the idea behind the proof of this lemma. Clearly, both G_1 and G_2 are spanning subgraphs of $(K_{2r}\times K_{2m})\circ\overline{K}_{ns}$. By Lemma 2.8, we can decompose $(C_{2r}^i\times K_{2m})\circ\overline{K}_{ns}$ into $(2m-1)ns+1$ linear forests; out of these $(2m-1)ns+1$ linear forests, see the proof of lemma 2.8, $2(m-1)ns$ are Hamilton paths, ns are linear forests and one is a matching M_i. Using these linear forests we shall obtain $(r-1)(2m-1)ns$ linear forests and a matching $\bigcup_{i=1}^{r-1}M_i$, in G_1 that is, $\ell a(G_1)=(r-1)(2m-1)ns+1$. Then we consider $G_2\cup\bigcup_{i=1}^{r-1}M_i$ and we prove that $\ell a(G_2\cup\bigcup_{i=1}^{r-1}M_i)=(m-1)ns+\lceil\frac{ns+1}{2}\rceil$. Thus we have $\ell a((K_{2r}\times K_{2m})\circ\overline{K}_{ns})=(r-1)(2m-1)ns+(m-1)ns+\lceil\frac{ns+1}{2}\rceil=\left\lceil\frac{\Delta((K_{2r}\times K_{2m})\circ\overline{K}_{ns})+1}{2}\right\rceil$.

Next we shall proceed to complete the proof of this lemma.



Each parallel line represents the subgraph isomorphic to $(K_2 \times K_{2m}) \circ \overline{K}_{ns}$. Also each normal line represents a 1-factor between the respective vertex subsets.

FIGURE 14. The graph $((F^{2r} \times K_{2m}) \circ \overline{K}_{ns}) \cup \bigcup_{i=1}^{r-1} M_i \cong Z_{2r,2m}^{ns}$ when $2r = 6$ and $2m = 8$.

If necessary, relabel the vertices of $C_{2r}^i, 1 \leq i \leq r-1$, in $\mathscr{C}$, so that the edge $E(M^{2r} \cap C_{2r}^i) = x_{2r-1}x_0$. Now each of the graphs $(C_{2r}^i \times K_{2m}) \circ \overline{K}_{ns}, 1 \leq i \leq r-1$, can be decomposed into $2(m-1)ns$ Hamilton paths, ns linear forests and a matching M_i by the proof of Lemma 2.8. Consequently, $\bigcup_{i=1}^{r-1} (C_{2r}^i \times K_{2m}) \circ \overline{K}_{ns} \subset (K_{2r} \times K_{2m}) \circ \overline{K}_{ns}$ has $(r-1)(2(m-1))ns$ Hamilton paths, $(r-1)ns$ linear forests and another linear forest, a matching $\bigcup_{i=1}^{r-1} M_i$, of $(K_{2r} \times K_{2m}) \circ \overline{K}_{ns}$; $\bigcup_{i=1}^{r-1} M_i$ is a matching because M^{2r} is a matching in K_{2r}.

From the proof of Lemma 2.8, it is clear that the matching M_i of $(C_{2r}^i \times K_{2m}) \circ \overline{K}_{ns}$ is contained in $(e_i \times K_{2m}) \circ \overline{K}_{ns}$, where e_i is the only edge in $C_{2r}^i \cap M^{2r}$. As $M^{2r} \cup F^{2r}$ is a Hamilton path of K_{2r}, the graph $((F^{2r} \times K_{2m}) \circ \overline{K}_{ns}) \cup (\bigcup_{i=1}^{r-1} M_i)$ is connected and it has the structure as shown in Figure 14.

It is an easy observation that $((F^{2r} \times K_{2m}) \circ \overline{K}_{ns}) \cup \bigcup_{i=1}^{r-1} M_i \cong Z_{2r,2m}^{ns}$. But $Z_{2r,2m}^{ns}$ has a decomposition into $(m-1)ns + \lceil \frac{ns+1}{2} \rceil$ linear forests, by Lemma 2.10.

Thus totally we have $(r-1)(2m-1)ns + (m-1)ns + \lceil \frac{ns+1}{2} \rceil = \left\lceil \frac{\Delta((K_{2r} \times K_{2m}) \circ \overline{K}_{ns})+1}{2} \right\rceil$, and hence LAC is true for the graph $(K_{2r} \times K_{2m}) \circ \overline{K}_{ns}$.

This completes the proof of the lemma. □

Lemma 2.12. *For $r \geq 3, m \geq 1$ and $ns \neq 2, 6$, the graph $(C_r \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n)$ can be decomposed into $2mns$ Hamilton paths and a matching, that is, the LAC is true for $(C_r \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n)$.*

Proof. If $n = s = 1$, then the proof follows from Lemma 2.3 and hence we assume that $n, s > 1$. Let $C_r = x_0 x_1 \ldots x_{r-1} x_0$. Clearly, $(C_r \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n) \cong (C_r \times K_{2m+1}) \circ \overline{K}_{ns}$. The Hamilton cycle decomposition of $C_r \times K_{2m+1}$, described in Lemmas 2.5 and 2.13 of [12], is orthogonal to the matching R'' (see the description of R'' at the beginning of this section and Figure 2(b)) constructed in [13]. Clearly, $E(C_r \times K_{2m+1}) = \cup_{i=0}^{r-1} \{\cup_{j=1}^{2m} F_j(X_i, X_{i+1})\}$, where the subscript of X is taken modulo r. Let $\mathscr{H} = \{H_1, H_2, \ldots, H_{2m}\}$ be the Hamilton cycle decomposition of $C_r \times K_{2m+1}$ such that the matching R'' is orthogonal to the Hamilton cycle decomposition $\mathscr{H}$ of $C_r \times K_{2m+1}$, where

$$\begin{aligned} R'' =& \{x_{r-1,0}x_{0,1}, x_{r-1,1}x_{0,2}, x_{r-1,2}x_{0,3}, \ldots, x_{r-1,2m-3}x_{0,2m-2}, \\ & x_{r-1,2m-2}x_{0,2m-1}, x_{r-1,2m-1}x_{0,0}\} \\ =& \oplus_{i=0}^{2m-2}\{x_{r-1,i}x_{0,i+1}\} \oplus \{x_{r-1,2m-1}x_{0,0}\}, \end{aligned}$$

see Lemma 2.11 of [13].

Note that $R'' \subset \langle X_{r-1} \cup X_0 \rangle \subset C_r \times K_{2m+1}$. Hence $(\bigcup_{i=1}^{2m} H_i) - R'' = (\bigcup_{i=1}^{2m} H_i')$, where H_i' is the Hamilton path arises out of the deletion of an edge of R'' in H_i.

Now $(C_r \times K_{2m+1}) \circ \overline{K}_{ns} = (H_1 \oplus H_2 \oplus \ldots \oplus H_{2m}) \circ \overline{K}_{ns} = \bigoplus_{i=1}^{2m} (H_i \circ \overline{K}_{ns})$. For $1 \leq i \leq 2m$, fix the unique edge of $H_i \cap R''$. If $H_i \cap R'' = x_{a,c}x_{b,d}$, then $H_i \circ \overline{K}_{ns}$ has a decomposition into ns Hamilton cycles and there is a matching in $\langle X_{a,c} \cup X_{b,d} \rangle$ which is orthogonal to these ns Hamilton cycles, by Lemma 2.7. We denote this matching by $M_i, 1 \leq i \leq 2m$, and let $M = \bigcup_{i=1}^{2m} M_i$, which is a matching as R'' is a matching in $C_r \times K_{2m+1}$. Thus we have decomposed $((C_r \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n))$ into $2mns$ Hamilton paths and a matching M.

This completes the proof of the lemma. □

Lemma 2.13. *For $r \geq 2, m \geq 1$ and $ns \neq 2, 6$, LAC is true for the graph $(K_{2r} \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n)$.*

Proof. If $n = s = 1$, then the proof follows from Lemma 2.4 and hence we assume that $n, s > 1$. Observe that the graph $(K_{2r} \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n) \cong (K_{2r} \times K_{2m+1}) \circ \overline{K}_{ns}$. The graph K_{2r} has a decomposition into $r - 1$ Hamilton cycles $\mathscr{C} = \{C_{2r}^1, C_{2r}^2, \ldots, C_{2r}^{r-1}\}$ and a perfect matching $F^{2r} = \{x_0x_1, x_2x_3, x_4x_5, \ldots, x_{2r-2}x_{2r-1}\}$. Further, there exists a matching M^{2r} of K_{2r} such that M^{2r} is orthogonal to the Hamilton cycles in $\mathscr{C}$ and $M^{2r} \cup F^{2r}$ is a Hamilton path of K_{2r}, by Lemma 2.2 of [13]. Thus

(2.2)

$$\begin{aligned}(K_{2r} \times K_{2m+1}) \circ \overline{K}_{ns} &= ((C^1_{2r} \oplus C^2_{2r} \oplus \ldots \oplus C^{r-1}_{2r} \oplus F^{2r}) \times K_{2m+1}) \circ \overline{K}_{ns} \\ &= \Big(\Big(\oplus_{i=1}^{r-1} (C^i_{2r} \times K_{2m+1})\Big) \circ \overline{K}_{ns}\Big) \\ &\quad \oplus \Big((F^{2r} \times K_{2m+1}) \circ \overline{K}_{ns}\Big) \\ &= G_1 \oplus G_2\end{aligned}$$

If necessary, relabel the vertices of $C^i_{2r}, 1 \leq i \leq r-1$, in $\mathscr{C}$, so that the unique edge of $E(M^{2r} \cap C^i_{2r}) = x_{2r-1}x_0$. Now each $(C^i_{2r} \times K_{2m+1}) \circ \overline{K}_{ns}, 1 \leq i \leq r-1$, can be decomposed into $2mns$ Hamilton paths and a matching M_i as in the proof of Lemma 2.12. Consequently, $\bigcup_{i=1}^{r-1}(C^i_{2r} \times K_{2m+1}) \circ \overline{K}_{ns} \subset (K_{2r} \times K_{2m+1}) \circ \overline{K}_{ns}$, has $(r-1)2mns$ Hamilton paths and a matching $\bigcup_{i=1}^{r-1} M_i$, of G_1; $\bigcup_{i=1}^{r-1} M_i$ is a matching of G_1 as M^{2r} is a matching of K_{2r}.

The set of edges of $(K_{2r} \times K_{2m+1}) \circ \overline{K}_{ns}$, which are not on these $(r-1)2mns$ Hamilton paths are the edges of $((F^{2r} \times K_{2m+1}) \circ \overline{K}_{ns}) \cup \bigcup_{i=1}^{r-1} M_i$. But $F^{2r} \times K_{2m+1} = \bigoplus_{j=1}^{r}(K_2 \times K_{2m+1})$. Let $X = \{x_{0,k} | 0 \leq k \leq 2m\}$ and $Y = \{x_{1,k} | 0 \leq k \leq 2m\}$ be the bipartition of $K_2 \times K_{2m+1}$. Now $\mathscr{H} = \{H_1, H_2, \ldots, H_m\}$ is a Hamilton cycle decomposition of $K_2 \times K_{2m+1}$, where $H_i = F_{2i-1}(X,Y) \cup F_{2i}(X,Y), 1 \leq i \leq m$. As $M' = \{x_{0,2i-1}x_{1,2(2i-1)} | 1 \leq i \leq m\}$ contains the edges of distinct odd jumps, M' is orthogonal to the Hamilton cycles in $\mathscr{H}$. The graph $((F^{2r} \times K_{2m+1}) \circ \overline{K}_{ns}) \cong \bigoplus_{j=1}^{r} \big((K_2 \times K_{2m+1}) \circ \overline{K}_{ns}\big)$. But $(K_2 \times K_{2m+1}) \circ \overline{K}_{ns} = (\bigoplus_{i=1}^{m} H_i) \circ \overline{K}_{ns} = \bigoplus_{i=1}^{m} \big(H_i \circ \overline{K}_{ns}\big)$. For each $i, 1 \leq i \leq m$, fix the unique edge $x_{0,2i-1}x_{1,2(2i-1)}$ of M' in H_i, that is, $H_i \cap M'$. Now $H_i \circ \overline{K}_{ns}$ has decomposition into Hamilton paths and a matching in $\langle X_{0,2i-1} \cup X_{1,2(2i-1)} \rangle$ of $H_i \circ \overline{K}_{ns}$, by Lemma 2.7. Hence we have mns linear forests of $(K_2 \times K_{2m+1}) \circ \overline{K}_{ns} \subset (K_{2r} \times K_{2m+1}) \circ \overline{K}_{ns}$. It is not difficult to check that the edges which are not in these mns linear forests of $(K_2 \times K_{2m+1}) \circ \overline{K}_{ns}$ together with $\bigcup_{i=1}^{r-1} M_i$ is a last linear forest, since $M^{2r} \cup F^{2r}$ is a Hamilton path of K_{2r}. Thus we have a decomposition of $(K_{2r} \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n)$ into $2rmns - mns + 1$ linear forests.

This completes the proof of the lemma. □

Lemma 2.14. *For $r, m \geq 1$ and $ns \neq 2, 6$, LAC is true for the graph $(K_{2r+1} \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n)$.*

Proof. If $n = s = 1$, then the proof follows from Lemma 2.5 and hence we assume that $n, s > 1$. Clearly, $(K_{2r+1} \circ \overline{K}_s) \times (K_{2m+1} \circ \overline{K}_n) \cong (K_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}$. In [13], it is shown that K_{2r+1} has a Hamilton cycle decomposition $\mathscr{C} = \{C^1_{2r+1}, C^2_{2r+1}, \ldots, C^r_{2r+1}\}$ and a path P of length r such that P is orthogonal to the Hamilton cycles in $\mathscr{C}$. Now

(2.3)

$$(K_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns} = ((C^1_{2r+1} \oplus C^2_{2r+1} \oplus \ldots \oplus C^r_{2r+1}) \times K_{2m+1}) \circ \overline{K}_{ns}$$
$$= \Big(\Big(\oplus_{i=1}^{r} (C^i_{2r+1} \times K_{2m+1})\Big) \circ \overline{K}_{ns}\Big)$$

If necessary, relabel the vertices of $C^i_{2r+1}, 1 \leq i \leq r$, in $\mathscr{C}$ so that the edge $E(P \cap C^i_{2r+1}) = x_{2r}x_0$. Now each of the graphs $(C^i_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}, 1 \leq i \leq r$, can be decomposed into $2mns$ Hamilton paths and a matching M_i as in the proof of Lemma 2.12. Consequently, $\bigcup_{i=1}^{r}(C^i_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}$ has $2rmns$ Hamilton paths and r matchings, $M_i, 1 \leq i \leq r$, of $(K_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}$. Clearly, $\bigcup_{i=1}^{r} M_i$ is a linear forest as P is a path of K_{2r+1}. Thus totally we have $2rmns + 1 = \left\lceil \frac{(\Delta(K_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}) + 1}{2} \right\rceil$ linear forests in $(K_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}$. Hence LAC is true for the graph $(K_{2r+1} \times K_{2m+1}) \circ \overline{K}_{ns}$.

This completes the proof of the lemma. □

Proof. Proof of Theorem 1.1. The proof follows directly from Lemmas 2.11, 2.13 and 2.14. □

As an immediate consequence, we have the following

Corollary 2.15. *Let $r, m \geq 3$ and $ns \neq 2, 6$. If H is a subgraph of $(K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n)$ such that $\Delta(H) = \Delta((K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n))$, then LAC is true for H.*

Corollary 2.16. *Let $1 \leq r \leq s, 1 \leq m \leq n$ and $ns \neq 2, 6$. If $K_{r,s,s,\ldots,s}$ (resp. $K_{m,n,n,\ldots,n}$) is a complete multipartite graph with $\ell \geq 2$ (resp. $\ell' \geq 2$) parts of size s (resp. n) and one part of size r (resp. m), then LAC is true for the graph $K_{r,s,s,\ldots,s} \times K_{m,n,n,\ldots,n}$.*

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