Contributions to Discrete Mathematics

Volume 20, Number 2, Pages 438–455
ISSN 1715-0868

MORE ALGORITHMIC RESULTS FOR PROBLEMS OF SPREAD OF INFLUENCE IN EDGE-WEIGHTED GRAPHS WITH AND WITHOUT INCENTIVES

SIAVASH ASKARI AND MANOUCHEHR ZAKER

ABSTRACT. Many phenomena in real-world social networks are interpreted as the spread of influence between activated and non-activated elements within the network. These phenomena are formulated by combinatorial graphs, where vertices represent the elements and edges represent social ties between elements. A main problem is to study important subsets of elements (target sets or dynamic monopolies) such that their activation spreads to the entire network. In edge-weighted networks, the influence between two adjacent vertices depends on the weight of their edge. In models with incentives, the main problem is to minimize the total amount of incentives (called optimal target vectors) that can be offered to vertices such that some vertices are activated and their activation spreads to the whole network. Algorithmic study of target sets and vectors is a hot research field. We prove an inapproximability result for optimal target sets in edge-weighted networks, even for complete graphs. Some other hardness and polynomial time results are presented for optimal target vectors and degenerate threshold assignments in edge-weighted networks. Lastly, we obtain a hardness result for target sets in edge-weighted tournaments.

1. INTRODUCTION AND RELATED MODELS

Spread of influence is a process in which individuals in a virtual or real-world community change their opinions or any kind of influence through communication and interaction with each other. Various phenomena are spreading in real and virtual social networks, where the connected members of the network are affected. Adoption of new economic products by word-of-mouth communication is one example. Also, in elections, people usually decide whether or not to vote based on the influence of people in their neighborhood. Viral marketing is another phenomenon that refers to the dissemination of information about products, behaviors, and its acceptance

This work is licensed under a Creative Commons "Attribution-NoDerivatives 4.0 International" license.

Received by the editors July 24, 2024, and in revised form October 15, 2024.
2000 *Mathematics Subject Classification.* 05C69, 05C85, 68Q25, 91D30.
Key words and phrases. Social graph; target set selection; optimal target vector; dynamic monopoly; spread of influence; edge-weighted graphs.

by people in the community. Therefore, the ability to control the spread of these phenomena is economically and politically desirable, which has created many optimization problems. Let $G = (V, E)$ be a graph representing a social network in which vertices represent the network members and edges indicate the social tie or relationship between the members of the network. We denote the set of neighbors and the vertex degree of every vertex v by $N(v)$ and $d_G(v) = |N(v)|$, respectively. Throughout the paper, by a threshold assignment for an underlying network G, we mean any function $\tau : V \longrightarrow \mathbb{N}$ which assigns thresholds to the vertices such that $1 \leq \tau(v) \leq d_G(v)$, for all $v \in V$. The value $\tau(v)$ indicates the hardness of susceptibility or influenceability of vertex v in front of an influence.

The activation process corresponding to (G, τ) is defined as follows. Firstly, a subset A_0 of vertices in G is activated. Denote by A_t the set of active vertices in each round t. Then a vertex $x \in V(G) \setminus A_t$ becomes active in round $t+1$ if and only if vertex x has at least $\tau(x)$ neighbors in A_t. Note that once a vertex is activated, it remains active until the end of the process. Such a subset A_0 is called the target set of the dynamic monopoly in (G, τ) if it activates the entire graph. The smallest cardinality of dynamic monopolies in (G, τ) is denoted by $dyn(G, \tau)$. The target set selection problem (TSS) is a decision problem which, for any instance (G, τ) and integer k, asks whether $dyn(G, \tau) \leq k$. Target set selection and dynamic monopolies have been investigated by various authors [1, 3, 4, 11, 15, 18, 20, 23]. Many algorithmic and hardness results for TSS were obtained in [4, 5, 7, 14]. By TSS($d = 3, t \in \{1, 2\}$) they mean the TSS problem restricted to regular graphs of degree 3, where all thresholds are 1 or 2. The following is Theorem 1.8 in [14].

Theorem 1.1 ([14])**.** *The* TSS($d = 3, t \in \{1, 2\}$) *problem cannot be approximated within the ratio of* $\mathcal{O}(2^{\log^{1-\epsilon} n})$ *for any fixed constant* $\epsilon > 0$*, unless* $\mathsf{P} = \mathsf{NP}$.

1.1. **Target set selection in edge-weighted networks.** Two different weighted versions of TSS were investigated in the literature. Models with weighted vertices and with weighted edges were studied in [22] and [6], respectively. An edge $e = uv$ with weight $w(e)$ means that the amount of influence between u and v is $w(e)$. Networks with weighted edges are more realistic than unweighted networks. In the case that the underlying network is edge-weighted, by ω we always mean a weight function which assigns a positive rational number to each edge of G. Irrational weights are not required since they can be approximated by rational numbers, but sometimes we insist that the weights are positive rational numbers. The spread of influence in this case naturally depends on the weight of the edges. The formal definition, presented in [25] is given in the following.

Let a triple (G, ω, τ) be given, where G is a simple graph and τ and ω are threshold and weight functions for the edges of G, respectively. The activation process corresponding to (G, ω, τ) is defined as follows. Initially, a subset A_0 of vertices in G is activated. A_t is the set of vertices activated in round t. Then a vertex $x \in V(G) \setminus A_t$ becomes active in round $t+1$ if and only if the following inequality holds, where $E_t(x)$ consists of all edges, say e, such that $e = xy$ for some vertex $y \in A_t$.

$$\sum_{e \in E_t(x)} \omega(e) \geq \tau(x).$$

Such a set A_0 is called a dynamic monopoly in (G, ω, τ) if by activating the vertices in A_0, the entire graph get activated. Denote the smallest cardinality of dynamic monopolies by $dyn(G, \omega, \tau)$ and call it an optimal target set of G.

The target set selection problem for edge-weighted graphs is defined as follows.
Name: Target Set Selection in (edge)-Weighted graphs (TSSW).
Instance: A triple (G, τ, ω).
Goal: Find $dyn(G, \omega, \tau)$.

1.2. **Target vectors and spread of influence with incentives.** To define the model with incentives, we start with a simple and practical example. Consider a company that wants to sell its products. The company may decide to use a discount on its products instead of offering a few products for free (as an award) in order to encourage people to buy these products. In fact, if members of the community are considered as vertices of the network, in this method, instead of focusing on $A_0 \subseteq V(G)$ as the target set, we target all vertices of the network so that the entire network is encouraged (or activated). In other words, we consider discounts as incentives on each vertex, and the goal is to activate the entire network by minimizing the sum of incentives assigned to the vertices. These ideas are introduced in [10]. Then Cordasco et al. [9] formalized the related model as follows.

Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ represents an underlying network and τ be an assignment of thresholds to the vertices of G. An assignment of incentives to the vertices of V is a vector $\mathbf{p} = (p(v_1), \ldots, p(v_n))$, where $p(v) \in \{0, 1, \ldots\}$ represents the amount of incentive we apply (consume) on $v \in V$. It can be assumed that $0 \leq p(v) \leq \tau(v) \leq d(v)$. The activation process in $G = (V, E)$, starting from the incentives $\mathbf{p}$ is as follows

$$\text{active}[\mathbf{p}, 0] = \{v \mid \ p(v) \geq \tau(v)\}.$$

Then, for all $t \geq 1$ define:

$$\text{active}[\mathbf{p}, t] = \text{active}[\mathbf{p}, t-1] \cup \{u : |N(v) \cap \text{active}[\mathbf{p}, t-1]| \geq \tau(u) - p(v)\}.$$

A vertex v is activated at round $t > 0$ if $v \in \text{active}[\mathbf{p}, t] \setminus \text{active}[\mathbf{p}, t-1]$. An assignment of incentives $\mathbf{p}$ is called a target vector whenever the activation process activates the entire network vertices with this assignment, that is, $\text{active}[\mathbf{p}, t] = V$ for some $t \geq 0$. The size of the incentive assignment $\mathbf{p} : V \longrightarrow \mathbb{N} \cup \{0\}$ is given by $\sum_{v \in V} p(v)$. A target vector of the minimum size is called an optimal target vector and is denoted by $\mathbf{p}^*$. The target set selection with incentives or optimal target vector problem is as follows.

Name: Optimal Target Vector (OTV).
Instance: A graph $G = (V, E)$ with a threshold assignment $\tau : V \longrightarrow \mathbb{N}$.
Goal: Find a target vector $\mathbf{p}$ with the minimum possible value $\mathbf{p}(V) = \sum_{v \in V} p(v)$.

Comprehensive studies have focused on this problem with different titles and obtained many results [8, 9, 12, 16, 21]. Using the result obtained by Chen [4], Cordasco et al. proved that OTV cannot be approximated within a ratio of $O(2^{\log^{1-\varepsilon} n})$, for any fixed $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{DTIME}\left(n^{\mathrm{polylog}(n)}\right)$ [9]. Feige and Kogan proved that OTV has a polynomial time solution if the threshold of any vertex in the input graph is 1 or $d(v)$, and or $\tau(v) \in \{d(v) - 1, d(v)\}$ [13]. OTV is also solvable in polynomial time for complete graphs, trees, and cycles [9]. OTV has also been investigated by Günneç et al. for bipartite graphs and directed graphs and proved to be solvable for directed acyclic graphs in polynomial time [16, 17].

We know that networks with weighted edges are more realistic in the real-world networks. For example, when advertising new products, influential people have a greater activation effect than ordinary people. Therefore, we quantify these influences as the weight of the edges between participants in the community so that the spread of influence depends on the weight of the edges. In the setting of Günneç et al. [16] (under the name of least-cost influence maximization problem) influence between any two vertices u and v depends on the weight of the edge $e = uv$. In the following, using the model of target set selection with incentives presented in [9] and the TSSW model given by the second author [25], we present our model for the spread of influence with incentives in edge-weighted graphs.

Consider a quadratic $(G, \omega, \tau, \mathbf{p})$ in which G is a simple graph with a weight function $\omega : E(G) \to (0, \infty)$ and a threshold assignment $\tau : V \to [0, \infty)$ and an incentive assignment $\mathbf{p} : V \to [0, \infty)$. The activation process corresponding to $(G, \omega, \tau, \mathbf{p})$ is defined as follows. Denote by $p(v)$ the incentive received by every vertex v. Initially, a subset A_0 of vertices in G is activated. Precisely, $A_0 = \{v : p(v) \geq \tau(v)\}$. Define A_t as a set of vertices activated in round t. Then a vertex $x \in V(G) \setminus A_t$ becomes active in round $t+1$ if and only if the following inequality holds, where $E_t(x)$ consists of all edges say e such that $e = xy$ for some vertex $y \in A_t$.

$$\sum_{e \in E_t(x)} \omega(e) + p(x) \geq \tau(x).$$

Definition 1.2. *Given (G, ω, τ), by a target vector $\mathbf{p}$, we mean an incentive assignment $\mathbf{p}$ such that the activation process corresponding to $(G, \omega, \tau, \mathbf{p})$ activates the whole graph G.*

The weighted version of OTV for networks with weighted edges, denoted by OTVW, is the following.
Name: Optimal Target Vector in Weighted Graphs (OTVW).
Instance: A triple (G, τ, ω), τ a threshold assignment and ω a weight function.
Goal: Find a target vector $\mathbf{p}$ which minimizes $\mathbf{p}(V) = \sum_{v \in V} p(v)$.

1.3. **Degenerate threshold functions.** Degenerate threshold functions were defined by Feige and Kogan [13] as follows. Let G be a simple undirected graph. A threshold function τ is a degenerate assignment for vertices of G if in every induced subgraph H of G, there exists a vertex $x \in V(H)$ such that $\tau(x) \geq d_H(x)$. This notion is similar to the concept of generalized degeneracy defined in [24]. Given $\kappa : V(G) \to \mathbb{N} \cup \{0\}$, a graph G is called κ-degenerate if the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that $d_{G[v_1,\ldots,v_i]}(v_i) \leq \kappa(v_i)$, for any $i \in \{1, \ldots, n\}$. It was proved in [24] that a set D in (G, τ) is a target set if and only if $V(G) \setminus D$ is κ-degenerate, where $\kappa(v) = d_G(v) - \tau(v)$. The TSS problem when the threshold function is degenerate is denoted by TSS(degenerate). The first complexity results concerning degenerate assignments were obtained in [13]. We generalize the idea for edge-weighted graphs.

Definition 1.3. *Given a triple (G, ω, τ), where G is a simple graph, a threshold function τ for the vertices of G is called degenerate if in every induced subgraph H of G, there exists a vertex $x \in V(H)$ such that $\tau(x) \geq \sum_{e \in E(x,H)} \omega(e)$, where $E(x, H)$ contains all edges of H, incident to x. The TSSW problem for (G, ω, τ), in which τ is a degenerate threshold function, is denoted by* TSSW*(degenerate).*

Feige and Kogan obtained an approximation algorithm for TSS(degenerate) [14]. By OTVW(degenerate), we mean the problem OTVW such that the threshold functions in its input (G, ω, τ) are restricted to degenerate threshold functions. The following result is derived routinely.

Proposition 1.4. *Given (G, ω, τ) on n vertices, where τ is a degenerate threshold assignment, then there is a degeneracy ordering of its vertices R : $u_1, u_2, \ldots, u_n$ such that for each $1 \leq i \leq n$, $\tau(u_i) \geq \sum_{e \in E(u_i,R)} \omega(e)$, where $E(u_i, R)$ consists of all edges between u_i and $\{u_1, u_2, \ldots, u_{i-1}\}$. Moreover, the ordering R can be obtained using $2nm$ additions and comparisons, where $m = |E(G)|$.*

Proof. By Definition 1.3, there exists a vertex $u_n \in V(G)$ such that $\tau(u_n) \geq \sum_{e \in E(u_n,G)} \omega(e) = \sum_{e \in E(u_n,R)} \omega(e)$. By applying Definition 1.3 for $G - u_n$, we obtain a next vertex u_{n-1} such that $\tau(u_{n-1}) \geq \sum_{e \in E(u_{n-1},G-u_n)} \omega(e) = \sum_{e \in E(u_{n-1},R)} \omega(e)$. By repeating this method for $G \setminus \{u_n, u_{n-1}\}$ we obtain a vertex u_{n-2} satisfying $\tau(u_{n-2}) \geq \sum_{e \in E(u_{n-1},R)} \omega(e)$. The other desired vertices $u_{n-3}, \ldots, u_2, u_1$ are obtained using the same method. Note that $E(u_1, R) = \emptyset$.

To determine the complexity of obtaining R, note that to find the first vertex u_n we need to search for a vertex v satisfying $\tau(v) \geq \sum_{e \in E(G): v \in e} \omega(e)$. Clearly, this can be done by exactly $deg_G(v)$ additions and comparison. Hence, u_n is obtained by $2|E(G)|$ time steps. We conclude that the remaining vertices in R are obtained by performing $2nm$ additions and comparisons. □

1.4. **The paper outline and a table of complexity problems.** The outline of the paper is as follows. Section 2 devotes to the problems TSSW and TSSW(degenerate). We first prove in Proposition 2.1 that for any $\epsilon > 0$, TSSW when the underlying graph is complete graph does not admit polynomial time approximation algorithm within ratio $\mathcal{O}(2^{\log^{1-\epsilon} n})$, unless $\mathsf{NP} = \mathsf{P}$. Next, we present an approximative algorithm and prove in Theorem 2.2 that the algorithm returns a target set of size at most $(\tau_{max}/c)\ OPT(G)$, where τ is a degenerate threshold assignment and c is defined in the algorithm. It is also proved that TSSW(degenerate, complete) is NP-complete. Section 3 is devoted to OTVW and OTVW(degenerate). We prove in Proposition 3.2 that OTVW(degenerate) can be solved in polynomial time. OTVW has polynomial time solutions for two restricted cases of threshold assignments (Theorems 3.4, 3.5). The last section is devoted to directed graphs. We prove in Proposition 4.1 that TSSWD (a directed version of TSSW) is NP-complete even for tournaments with positive edge weights, where by a tournament we mean any edge orientation of a complete graph. Figure 1 summarizes the complexity results concerning the decision problems discussed in this paper, where we use a short notation † to denote the "degenerate" version of the problem (i.e. the threshold assignments in the problem are restricted to degenerate ones). Hence, for example OTV(†) = OTV(degenerate).

Problem	Hardness	References
TSS	Inapproximable unless P = NP	[4, 11]
OTV	NP-complete	[9]
OTVW	NP-complete for complete graphs	[2]
TSSW	Inapproximable for complete graphs unless P = NP	This paper
OTV(†)	Polynomial-time	[13]
TSS(†)	NP-complete	[14]
TSSW(†)	NP-complete for complete graphs	This paper
OTVW(†)	Polynomial-time	This paper
OTVW	Polynomial-time for restricted threshold assignments	This paper
TSSWD	NP-complete for directed tournaments	This paper

FIGURE 1. A table of problems discussed in the paper and their complexity status, where † stands for "degenerate".

2. Results for TSSW and TSSW(degenerate)

In the following, by TSSW(complete) we mean the problem TSSW restricted to edge-weighted complete graphs. Theorem 1.1 asserts that TSS($d = 3, t \in \{1, 2\}$) cannot be approximated within the ratio of $\mathcal{O}(2^{\log^{1-\epsilon} n})$ for any fixed constant $\epsilon > 0$, unless $\mathsf{P} = \mathsf{NP}$. We use this result to prove the same inapproximability result for TSSW(complete). This in particular shows that TSSW(complete) and then TSSW is NP-hard.

Proposition 2.1. *For any* $\epsilon > 0$*,* TSSW*(complete) does not admit polynomial time approximation algorithm within ratio* $\mathcal{O}(2^{\log^{1-\epsilon} n})$*, unless* $\mathsf{NP} = \mathsf{P}$.

Proof. We obtain a gap-preserving reduction from TSS($d = 3, t \in \{1, 2\}$) to TSSW(complete). Let (G, τ) be an instance of TSS($d = 3, t \in \{1, 2\}$) on n vertices. We obtain a complete graph K_n from G such that $V(K_n) = V(G)$ as follows. For each vertex $v \in V(K_n) = V(G)$ define $\tau'(v) = n\tau(v)$. For each edge $e \in E(K_n)$ define $\omega(e) = n$, if $e \in E(G)$ and $\omega(e) = 1$, if $e \notin E(G)$. We prove that any target set D for (G, τ) is also a target set in (K_n, τ', ω) and vice versa.

Assume that $D_0 \subseteq V(G)$ is a target set for (G, τ) and let $|D_0| = k$. Suppose that the activation process in G has started with D_0. Denote by D_i the set of vertices activated up to round i in this activation process. We show that D_0 is a target set for K_n as well. Let $v \in V(K_n) \setminus D_0$ be an arbitrary vertex. We prove that if v is activated in G in a round, say i, then v as a vertex in K_n becomes active in round i. Note that v receives $n|E(v, D_{i-1})| + |D_{i-1}| - |E(v, D_{i-1})|$ influence from its neighbors in K_n at round i, where $E(v, D_{i-1})$ consists of all edges $e = (v, u)$ in G such that $u \in D_{i-1}$. Hence, it is enough to prove the following inequality.

$$n|E(v, D_{i-1})| + |D_{i-1}| - |E(v, D_{i-1})| \geq \tau'(v) = n\tau(v). \tag{2.1}$$

Vertex v is activated in G at round i. Hence, $|E(v, D_{i-1})| \geq \tau(v)$ and

$$n|E(v, D_{i-1})| \geq n\tau(v) = \tau'(v).$$

Since $|D_{i-1}| - |E(v, D_{i-1})| \geq 0$, therefore inequality (2.1) holds.

To prove the reverse direction, let $W_0 \subseteq V(K_n)$ be a target set for (K_n, τ', ω). Set $|W_0| = k$. Suppose the activation process in K_n is started with W_0. The set of vertices activated up to round i is denoted by W_i, and t is the total number of activation rounds in this graph. We claim that W_0 is a target set for G. Let the activation process in G be started with W_0. We set $D_0 = W_0$ and denote the set of vertices activated up to round i in G by D_i. By induction on $0 < j \leq t$, we show that $D_j = W_j$. In other words, we show that if the arbitrary vertex v of K_n is activated in a round j, then it is activated in G in the same round. Suppose that $D_i = W_i$ for each $i < j \leq t$. Then $D_{j-1} = W_{j-1}$. Let $v \in W_j$ be an arbitrary vertex. We show that $v \in D_j$. Note that $|D_{j-1}| \leq n - 1$. Since v is activated in K_n at round j we have the following inequality, where $E(v, D_{j-1})$ consists of all edges $e = (v, u)$ in G such that $u \in D_{j-1}$.

$$n\tau(v) = \tau'(v) \leq n|E(v, D_{j-1})| + |D_{j-1}| - |E(v, D_{j-1})|. \quad (2.2)$$

Inequality 4.1 together with the clear inequality $|D_{j-1}| - |E(v, D_{j-1})| \leq n-2$ imply

$$\begin{aligned} n\tau(v) &\leq n|E(v, D_{j-1})| + (n-2) \\ \tau(v) - |E(v, D_{j-1})| &\leq \frac{n-2}{n} < 1 \\ \tau(v) - |E(v, D_{j-1})| &\leq 0 \\ \tau(v) &\leq |E(v, D_{j-1})|. \end{aligned}$$

Hence, v is activated in round j in G. Similarly, by induction on j, we show that $D_j \subseteq W_j$. Let v be an arbitrary vertex in D_j. Hence, $|E(v, D_{j-1})| \geq \tau(v)$ and $n|E(v, D_{j-1})| \geq n\tau(v)$. Also, since $|D_{j-1}| - |E(v, D_{j-1})| \geq 0$, therefore

$$n|E(v, D_{j-1})| + |D_{j-1}| - |E(v, D_{j-1})| \geq n\tau(v) = \tau'(v).$$

In other words, v in (K_n, ω) is activated in round j. Namely, $v \in W_j$. □

Let a triple (G, ω, τ) be given. Recall from subsection 1.3 that a threshold function τ is called degenerate if in every induced subgraph H of G, there exists a vertex $x \in V(H)$ such that $\tau(x) \geq \sum_{e \in E(x,H)} \omega(e)$, where $E(x, H)$ contains all edges of H incident to x. Recall also that TSSW(degenerate) is a subproblem of TSSW, where the threshold functions are degenerate.

Feige and Kogan in [14] obtained an approximation algorithm with approximate ratio τ_{max} for TSS(degenerate), where $\tau_{max} = \max\{\tau(v) : v \in V(G)\}$. We obtain an approximation algorithm for TSSW(degenerate) by generalizing their method. Then we show that the problem is NP-complete even for complete graphs. Given (G, ω, τ), define $\tau_{max} = \max\{\tau(v) : v \in V(G)\}$ and let $OPT(G)$ denote the size of an optimal target set in (G, ω, τ). Define also $c = \min_{u \in R}\{\tau(u) - \sum_{e \in E(u,R)} \omega(e) : \tau(u) > \sum_{e \in E(u,R)} \omega(e)\}$, where R is a degeneracy ordering of $V(G)$ corresponding to τ obtained in Proposition 1.4 as $u_1, \ldots, u_n$. Note that $E(u_1, R) = \emptyset$ and $\sum_{e \in E(u,R)} \omega(e) = 0$ and hence $\tau(u_1) > \sum_{e \in E(u_1,R)} \omega(e)$. This proves that c is well-defined. By the proof of Proposition 1.4, all terms $\tau(u) - \sum_{e \in E(u,R)} \omega(e)$, $u \in R$ and hence the parameter c are determined using $2|V(G)| \times |E(G)|$ comparisons and additions. Obviously $\tau_{\max}/c \geq 1$.

Algorithm I.
Input: A triple (G, ω, τ) in which G is a simple graph on n vertices, ω is the weight function on E, and τ is a degenerate threshold assignment for V.
Output: A target set S of size at most $(\tau_{max}/c)\ OPT(G)$.
1. $S = \varnothing$
2. Let $R : u_1, u_2, \ldots, u_n$ be a degeneracy ordering of the vertices of G

3. For each $1 \leq i \leq n$, if $\tau(u_i) > \sum_{e\in E(u_i,R)}\omega(e)$ then $S \leftarrow S \cup \{u_i\}$
4. Return S

Theorem 2.2. *Given a triple (G, ω, τ) on n vertices, where τ is a degenerate threshold assignment. Then Algorithm I has running time $\mathcal{O}(n)$ and returns a target set of size at most $(\tau_{\max}/c)\ OPT(G)$.*

Proof. Let S be an output of Algorithm I. Set S is a target set for G, because for each $1 \leq i \leq n$, if vertex u_i is not selected by the algorithm, then according to the property of degeneracy ordering, we have $\tau(u_i) = \sum_{e\in E(u_i,R)}\omega(e)$. Therefore, the elements of S activate all vertices not selected by the algorithm, exactly based on the order in which they are in the degeneracy ordering. Let $|S| = s$ we show that $s \leq (\tau_{\max} \times OPT(G))/c$. Since R is degeneracy order by condition 3 of the algorithm and by Proposition 1.4, for each vertex u, $\tau(u) \geq \sum_{e\in E(u,R)}\omega(e)$, then

$$\sum_{i=1}^{n}\tau(u_i) \geq \sum_{u\in S}\left[\sum_{e\in E(u,R)}\omega(e) + (\tau(u) - \sum_{e\in E(u,R)}\omega(e))\right] + \sum_{u\in V(G)\setminus S}\left[\sum_{e\in E(u,R)}\omega(e)\right]$$

Recall that

$$\begin{aligned} c &= \min_{u\in R}\{(\tau(u) - \textstyle\sum_{e\in E(u,R)}\omega(e)) : \tau(u) - \textstyle\sum_{e\in E(u,R)}\omega(e) \neq 0\} \\ &= \min_{u\in S}\{\tau(u) - \textstyle\sum_{e\in E(u,R)}\omega(e)\}, \end{aligned}$$

hence we have

$$\begin{aligned} \sum_{i=1}^{n}\tau(u_i) &\geq \sum_{u\in S}\left[\sum_{e\in E(u,R)}\omega(e) + c\right] + \sum_{u\in V(G)\setminus S}\left[\sum_{e\in E(u,R)}\omega(e)\right] \\ &\geq sc + \sum_{u\in S}\left[\sum_{e\in E(u,R)}\omega(e)\right] + \sum_{u\in V(G)\setminus S}\left[\sum_{e\in E(u,R)}\omega(e)\right] \\ &= sc + \sum_{u\in V(G)}\left[\sum_{e\in E(u,R)}\omega(e)\right] = sc + \sum_{e\in E(G)}\omega(e). \end{aligned}$$

Therefore

$$\sum_{i=1}^{n}\tau(u_i) \geq sc + \sum_{e\in E(G)}\omega(e). \tag{2.3}$$

Let $S^\star$ be an optimal target set of cardinality $\mathrm{OPT}(G)$, then by an inequality in [1], we have $\sum_{u\in V(G)\setminus S^\star}\tau(u) \leq |E(G)|$. Therefore

$$\sum_{u\in V(G)\setminus S^\star} \tau(u) \leq \sum_{e\in E(G)} \omega(e). \tag{2.4}$$

By inequality (2.3) we have

$$\sum_{u\in S^\star} \tau(u) + \sum_{u\in V(G)\setminus S^\star} \tau(u) \geq sc + \sum_{e\in E(G)} \omega(e). \tag{2.5}$$

By inequalities (2.4) and (2.5), we conclude $sc \leq \sum_{u\in S^\star}\tau(u)$ and $s \leq \frac{\tau_{\max}}{c}\ \mathrm{OPT}(G)$. □

Concerning Theorem 2.2 we should mention a remark. When graph G is unweighted then all edges have weight 1 and then $\sum_{e\in E(u,R)}\omega(e) = d_R(u)$. Hence, for unweighted graphs the definition of c reduces to $c = \min_{u\in R}\{\tau(u) - d_R(u) : \tau(u) > d_R(u)\} \geq 1$. In this case, $(\tau_{\max}/c) \leq \tau_{\max}$. It follows that, depending on c, this ratio is better than the ratio $\tau_{\max}$ obtained by Feige and Kogan [14], but the guaranteed lower bound for c is 1, which results in the same ratio $\tau_{\max}$ for the problem.

In the following, we use a result from [25]. By a vertex cover in a graph G, we mean any subset $B \subseteq V(G)$ such that each edge e has at least one endpoint in B. Denote by $\beta(G)$ the minimum $|B|$, where B is a vertex cover in G. Given any triple (G, ω, τ), it was proved in Proposition 2.4 [25] that (G, ω, τ) has a dynamic monopoly (target set) of cardinality at most $\beta(G)$. Note that for any connected graph G, $\beta(G) \leq |V(G)| - 1$. It was proved in [14] that TSS(degenerate) is NP-complete. In the following, by TSSW(degenerate, complete) we mean the problem TSSW restricted to degenerate threshold assignments, where the underlying graph is complete. We show in the following that TSSW(degenerate, complete) is NP-complete. We need some information from the proof of Proposition 2.1. In the proof, corresponding to each instance (G, τ) of TSS on n vertices, we obtained a complete graph K_n from G such that $V(K_n) = V(G)$, a threshold assignment τ' such that $\tau'(v) = n\tau(v)$, $v \in V(K_n)$ and a weight assignment ω such that $\omega(e) = n$ for $e \in E(G)$ and $\omega(e) = 1$ for $e \notin E(G)$. Denote the triple (K_n, τ', ω) in Proposition 2.1 by H_n. Hence, Proposition 2.1 proves that every target set D for (G, τ) is a target set for H_n and vice versa.

Proposition 2.3. TSSW(*degenerate, complete*) *is* NP-*complete.*

Proof. Clearly TSSW(degenerate,complete) belongs to NP. We obtain a polynomial time reduction from TSS(degenerate) to TSSW(degenerate, complete). Let (G, τ) be an instance of TSS(degenerate) on vertex set $\{v_1, \ldots, v_n\}$. We make an instance $(K_{n+1}, \omega, \tau'')$ of TSSW(degenerate, complete) as follows. Add a new vertex v_{n+1} to G and connect it to every vertex in $V(G)$. We have $V(K_{k+1}) = V(G) \cup \{v_{n+1}\}$. Define τ'' as follows. For each $i \in \{1, \ldots, n\}$, set $\tau''(v_i) = n\tau(v_i) + n$ also $\tau''(v_{n+1}) = n^2$. We define weight assignment for the edges of K_{n+1} as follows. Define $\omega(v_i v_{n+1}) = n$, for each $i \in \{1, \ldots, n\}$. Then for any other edge $e \in E(K_{n+1})$, if $e \in E(G)$ then set $\omega(e) = n$, otherwise $\omega(e) = 1$.

We prove that τ'' is degenerate. Since τ is degenerate in G then by definition there exists an ordering of $V(G)$ such as $R : v_1, v_2, \ldots, v_n$ such that $\tau(v_i) \geq d_R(v_i)$, where $d_R(v_i)$ denotes the number of neighbors of v_i in $G[v_1, \ldots, v_i]$. It is easily seen that τ'' with ordering of vertices $R' : v_1, \ldots, v_n, v_{n+1}$ is degenerate. We only have to check the last vertex v_{n+1}.

We show in the following that (G, τ, k) is Yes-instance for TSS(degenerate) if and only if $(K_{n+1}, \omega, \tau'', k+1)$ is Yes-instance for TSSW(degenerate, complete). First, let D be a target set for (G, τ, k). We prove that $D \cup \{v_{n+1}\}$ is a target set in $(K_{n+1}, \omega, \tau'')$. In K_{n+1} after activating v_{n+1} the threshold of each vertex v_i, $1 \leq i \leq n$, is reduced by $\omega(v_i v_{k+1}) = n$. Hence, for each v_i, the new threshold of v_i is $\tau'(v_i) = \tau''(v_i) - n = n\tau(v_i)$. We have also $\omega(e) = n$ for $e \in E(G)$, otherwise $\omega(e) = 1$. We observe that (K_n, τ', ω) is identical to H_n defined before the proof. From the proof of Proposition 2.1, we know that D is a target set in H_n since it is a target set in (G, τ). This proves that $D \cup \{v_{n+1}\}$ is a target set in $(K_{n+1}, \omega, \tau'')$ of size $k + 1$.

Assume now that S is a target set of size $k + 1$ in $(K_{n+1}, \omega, \tau'')$, where $k \geq 0$. There are two possibilities.

Case 1: $v_{n+1} \notin S$.

In this case, since $\tau(v_{n+1}) = \sum_{i=1}^{n} \omega(v_i v_{n+1})$, then $v_i \in S$ for each i with $1 \leq i \leq n$. Hence, $k + 1 \geq n$ or $k \geq n - 1 \geq \beta(G)$. It follows that G has a target set of at most $\beta(G) \leq k$ vertices, as desired.

Case 2: $v_{n+1} \in S$.

In this case, we prove that $S' = S \setminus v_{n+1}$ is a target set in G. Clearly, $S' \subseteq V(G)$ is a target set for $H_n = (K_n, \tau', \omega)$. It follows from the comment before the proof that G admits a target set of size $|S'| = k$.

This completes the proof. □

3. Results for OTVW and OTVW(degenerate)

We first present a result for OTVW(degenerate). Feige and Kogan proved that OTV(degenerate) has a polynomial time solution [13]. We use the following result proved in Proposition 2 in [2].

Proposition 3.1 ([2]). *Let (G, ω, τ) be a weighted graph and $\mathbf{p}^*$ be an optimal target vector in G. Then*

$$\sum_{v \in V(G)} \mathbf{p}^*(v) \geq \sum_{v \in V(G)} \tau(v) - \sum_{e \in E(G)} \omega(e).$$

By generalizing the method of [13], we show that OTVW(degenerate) is solved in polynomial time.

Proposition 3.2. OTVW(*degenerate*) *can be solved in polynomial time.*

Proof. Let $G = (V, E), \omega, \tau)$ be given, where τ is degenerate. By Proposition 3.1

$$\sum_{v \in V(G)} \mathbf{p}^*(v) \geq \sum_{v \in V(G)} \tau(v) - \sum_{e \in E(G)} \omega(e). \tag{3.1}$$

Since τ is degenerate, we deduce from Proposition 1.4 that there exists an ordering of vertices in G such as $R : u_1, u_2, \cdots, u_n$ such that $\tau(u_i) \geq \sum_{e \in E(u_i,R)} \omega(e)$, for each $1 \leq i \leq n$. Now, starting from u_1 scan the vertices in R and for each $1 \leq i \leq n$, if $\tau(u_i) > \sum_{e \in E(u_i,R)} \omega(e)$ then by placing $\mathbf{p}^*(u_i) = \tau(u_i) - \sum_{e \in E(u_i,R)} \omega(e)$, vertex u_i is activated. Otherwise, u_i is activated by its previous neighbors in R. Thus, with this incentive assignment to vertices of G, the entire graph is activated. A target vector of cost $\sum_{i=1}^{n} \mathbf{p}^*(u_i) = \sum_{i=1}^{n} \tau(u_i) - \sum_{i=1}^{n} \left[\sum_{e \in E(u_i,R)} \omega(e)\right]$ is obtained. Since $\sum_{i=1}^{n} \left[\sum_{e \in E(u_i,R)} \omega(e)\right] = \sum_{e \in E(G)} \omega(e)$, the cost of the target vector is $\sum_{i=1}^{n} \tau(u_i) - \sum_{e \in E(G)} \omega(e)$. From inequality 3.1, we conclude that the obtained solution is optimal. □

It was proved in [13] that if we consider threshold assignments τ such that $\tau(v) \in \{d_G(v) - 1, d_G(v)\}$, for each vertex v of a graph G then OTV has a polynomial time solution. In Theorem 3.4 we generalize this result for weighted graphs. Note that if $\tau(v) = d_G(v)$ then TSS is equivalent to the minimum vertex cover problem and hence an NP-hard problem [25]. In a weighted graph (G, ω) define $\mu(G, \omega) = \min\{\omega(e) : e \in E(G)\}$.

Lemma 3.3. *Let (G, ω, τ) be given, where G is connected on n vertices. Assume that for each $u \in V(G)$, $\tau(u) \geq \sum_{e \in E(u,G)} \omega(e) - \mu$, where $\mu = \mu(G, \omega)$. Assume that there exists $x \in V(G)$ such that $\tau(x) \geq \sum_{e \in E(x,G)} \omega(e)$. Then τ is degenerate.*

Proof. We use an induction on n. The assertion holds trivially for $n \leq 2$. Let G be a weighted connected graph on n vertices, such that there is a vertex $x \in V(G)$ satisfying $\tau(x) \geq \sum_{e \in E(x,G)} \omega(e)$. Let H be a connected components in $G \setminus x$ and $z \in V(H)$ a neighbor of x in H. We have

$$\tau(z) \geq \sum_{e \in E(z,G)} \omega(e) - \mu \geq \sum_{e \in E(z,G)} \omega(e) - \omega(xz) = \sum_{e \in E(z,H)} \omega(e). \tag{3.2}$$

Therefore, H satisfies the induction hypothesis, and then τ is degenerate for H. τ is degenerate for any component of $G \setminus x$ and hence for G itself. □

Now, by Lemma 3.3 and Proposition 3.2, we prove the following theorem which generalizes the case OTV$(\tau(v) \in \{d_G(v) - 1, d_G(v)\})$.

Theorem 3.4.

$$\text{OTVW}\left(\tau(u) \in \left\{\sum_{e \in E(u,G)} \omega(e) - \mu(G, \omega),\ \sum_{e \in E(u,G)} \omega(e)\right\}\right)$$

can be solved in polynomial time.

Proof. Let (G, ω, τ) be an input of the problem, where G is a connected graph. Write $\mu = \mu(G, \omega)$. If there exists a vertex $x \in V(G)$ such that $\tau(x) \geq \sum_{e \in E(x,G)} \omega(e)$, then by Lemma 3.3, τ is degenerate assignment and the proof is complete by Proposition 3.2. Otherwise, we have $\tau(u) = \sum_{e \in E(u,G)} \omega(e) - \mu$, for each $u \in V(G)$. Let $\mathbf{p}^*$ be an optimal target vector for G and let w be the last vertex in the activation process corresponding to $\mathbf{p}^*$. Vertex w is activated by $\sum_{e \in E(u,G)} \omega(e) - \mu$ influence from its neighbors hence there exists an edge say e_0 incident to w, which is not used in the activation process. It follows that the sizes of OTV for G and $H = G \backslash e$ (with the same weights and thresholds) are equal. Also we have $\sum_{e \in E(u,G)} \omega(e) - \omega(e_0) \geq \tau(w)$. Then $\omega(e_0) = \mu$. Let $e_0 = uw$. Let H_w (resp. H_w) be the connected component of H containing w (resp. u). Note that possibly $H_u = H_w$. We have $\tau(w) \geq \sum_{e \in E(w,H_w)} \omega(e)$ and $\tau(u) \geq \sum_{e \in E(u,H_u)} \omega(e)$. Then the connected graphs H_u and H_w satisfy the conditions of Lemma 3.3, hence by this lemma τ is degenerate for H_u and H_w. Now by Proposition 3.2, OTVW(H_u) and OTVW(H_w) and hence OTVW(H) are determined in polynomial time. □

It was proved in [13] that if $\tau(v) \in \{1, d_G(v)\}$ for each vertex v of a graph G then OTV$(\tau(v) \in \{1, d_G(v)\})$ has polynomial time solution. By generalizing this result we prove in the following theorem that OTVW has polynomial time solution for graphs G, where for each $u \in V(G)$, $\tau(u) = \sum_{e \in E(u,G)} \omega(e)$ or $\tau(u) = \mu(G, \omega)$.

Theorem 3.5. OTVW $\left(\tau(u) \in \left\{\sum_{e \in E(u,G)} \omega(e),\ \mu(G, \omega)\right\}\right)$ *has a polynomial time solution.*

Proof. Assume that τ is such that for each $u \in V(G)$, $\tau(u) = \mu$ or $\tau(u) = \sum_{e \in E(u,G)} \omega(e)$. We partition the vertices of G into two sets V_1 and V_2 as follows. V_1 contains vertices whose threshold is μ and $V_2 = V(G) \setminus V_1$. Now, from G, we construct a weighted multigraph $\mathbb{M}$ as follows. For each connected component C of $G[V_1]$, we replace $V(C)$ by a single vertex c (representing the component C) with threshold μ. Let C' be a set consisting of all such vertices c. Define $V(\mathbb{M}) = C' \cup V_2$. We put no edge between any two vertices in C', i.e., C' forms an independent set in $\mathbb{M}$. For each vertex $x \in V_2$ in G and each vertex z in a connected component C of $G[V_1]$, if there is an edge between x and z in G with weight k, we place an edge of weight k between x and c in multigraph $\mathbb{M}$. Note that if x has f neighbors in C then the edge xc has multiplicity f in $\mathbb{M}$. We do the above process for each vertex of V_2 and each connected component of $G[V_1]$ to complete the construction of $\mathbb{M}$.

Next, we convert $\mathbb{M}$ into a simple graph F as follows. Instead of every edge xy of $\mathbb{M}$, we place a new vertex s with $\tau(s) = \omega(xy)$ and edges xs and sy with $\omega(xs) = \omega(sy) = \omega(xy)$. Denote by S the set of vertices added in this step. We have the partition $V(F) = C' \cup V_2 \cup S$. Recall that V_2

consists of all vertices u such that $\tau(u) = \sum_{e\in E(u,F)}\omega(e)$. Observe that for each $u \in V_2$, $\tau(u) = \sum_{e\in E(u,F)}\omega(e) = \sum_{e\in E(u,G)}\omega(e)$. Every vertex s in S has degree two and then $\sum_{e\in E(s,F)}\omega(e) = 2\tau(s)$. Note that C' contains any vertex of threshold μ in F.

Now, we organize an ordering R of the vertices in F in such a way that at the beginning of the order, the vertices of C', then the vertices of S, and finally the vertices of V_2 appear. Then $R : C', S, V_2$ is the order of the vertices in F from left to right. We show that R is a degeneracy ordering in F. We know that C' is an independent set in F and for each $v \in C'$, $\tau(v) = \mu$ and then $\tau(v) > \sum_{e\in E(v,R)}\omega(e) = 0$. Now we check the vertices of S. For each $s \in S$, only one edge say e_0 is between C' and s. We have $\tau(s) = \omega(e_0) = \sum_{e\in E(s,R)}\omega(e)$. Finally, for each $v \in V_2$ we have $\tau(v) = \sum_{e\in E(v,F)}\omega(e) \geq \sum_{e\in E(v,R)}\omega(e)$. Therefore, R is a degeneracy order of the vertices in F. We converted the input instance G of $OTVW\left(\tau(u) \in \left\{\sum_{e\in E(u,G)}\omega(e), \mu\right\}\right)$ into an instance F of $OTVW(degenerate)$. According to Proposition 3.2, there is an activation process on order R by which the optimal target vector for graph F is obtained.

Now, we show that the activation process according to R also results in an optimal target vector for G. Every vertex in C' represents a connected component in $G[V_1]$, where all vertices have threshold μ. Thus, in this activation process, by activating only one vertex from each connected component, the whole component is activated. By the above definition, instead of every edge $(x, y) \in E(\mathbb{M})$, where $x \in C'$ and $y \in V_2$, the vertex s with $\tau(s) = \omega(x, y)$ is placed, and two edges (x, s) and (s, y) are created with $\omega(x, s) = \omega(s, y) = \omega(x, y)$. Also, for each $s \in S$ we have $\tau(s) = \sum_{e\in E(s,R)}\omega(e)$. Thus, by activating the vertices of C', each vertex $s \in S$ that is placed instead of an edge (x, y) is activated without receiving incentives. Also, vertex s influences y as much as the weight of edge (x, y). In other words, to activate the vertices of V_2 in this activation process, we don't need the vertices of S. Now, due to the structure of F, for each vertex $v \in V_2$, $\tau(v) = \sum_{e\in E(v,F)}\omega(e) = \sum_{e\in E(v,G)}\omega(e)$. Thus, the activation process on order R assigns the same incentives to the vertices of V_2 for both F and G. We conclude that the activation process in which an optimal solution is obtained for F also gives the optimal solution for G. □

4. A result for weighted tournaments

A complete model for the spread of influence in networks is to consider bidirected graphs with positive weighted edges. Let $(\overleftrightarrow{G}, \omega)$ be a weighted bidirected graph in which for any two vertices u and v in $\overleftrightarrow{G}$, the influence of u on v (resp. the influence of v on u) is proportional to $\omega(uv)$ (resp. $\omega(vu)$). In real-world networks, the influence of two people on each other is not necessarily identical. For example, very influential people have more influence on their neighbors than ordinary people. In these situations we need to use bidirected edge-weighted graphs. Consider a triple $(\overleftrightarrow{G}, \omega, \tau)$, where τ assigns a threshold $\tau(u) \leq d^{in}(u)$ to each vertex u. The activation process corresponding to $(\overleftrightarrow{G}, \omega, \tau)$ is defined as follows. Initially, a subset A_0 of vertices in $\overleftrightarrow{G}$ is activated. A_t is the set of vertices activated in round t. Then a vertex $x \in V(\overleftrightarrow{G}) \setminus A_t$ becomes active in round $t+1$ if and only if $\sum_{e \in E_t^{in}(x)} \omega(e) \geq \tau(x)$, where $E_t^{in}(x)$ is the set of incoming edges from A_t to x.

Target sets, target vectors, and optimal target sets in $(\overleftrightarrow{G}, \omega, \tau)$ and any directed graph with weighted edges are defined similarly. It was proved in [2] that optimal target vectors in bidirected edge-weighted paths and cycles can be solved polynomially by dynamic programming algorithms. By TSSWD we mean the problem of determining the optimal size of target sets in directed or bidirected graphs. Note that every instance $(G = K_n, \omega, \tau)$ of TSSW(complete) can easily be transformed to a bidirected complete graph H. It is enough to replace each edge $e = uv$ of weight $\omega(uv)$ in G by two directed edges (u, v) and (v, u) with new weights $\omega'(u, v) = \omega'(v, u) = \omega(uv)$ to obtain H. It follows that TSSWD is NP-complete for bidirected complete graphs. This problem is nontrivial if we only consider unilateral directed graphs, where between any two vertices u and v, either there is no edge from u to v or there is no edge from v to u. It was proved in [19] that to determine optimal target sets in bilateral directed graphs with constant thresholds 2 is NP-complete. For simplicity, denote the latter problem by TSSD($\tau = 2$). Clearly, a directed counterpart of TSSW(complete) problem is to consider tournaments. A tournament on n vertices is any edge orientation of the simple undirected complete graph K_n. Define similarly TSSWD(tournament) as directed counterpart of TSSW(complete). By combining the proof idea of Proposition 2.1 and NP-completes of optimal target sets in bilateral directed graphs in [19] we obtain the following result.

Proposition 4.1. TSSWD(*tournament*) *i.e.* TSSWD *for tournaments is* NP-*complete, where the weight of every directed edge is positive.*

Proof. We obtain a reduction from TSSD($\tau = 2$) to TSSWD(tournament). Let $(\overrightarrow{G}, \tau = 2)$ be an instance of TSSD($\tau = 2$) on n vertices. We obtain a tournament T_n from $\overrightarrow{G}$ such that $V(T_n) = V(\overrightarrow{G})$ as follows. For each vertex $v \in V(T_n) = V(\overrightarrow{G})$ define $\tau'(v) = 2n$. For each edge $e \in E(T_n)$ define $\omega(e) = n$, if $e \in E(\overrightarrow{G})$ and $\omega(e) = 1$, if $e \notin E(\overrightarrow{G})$. We arbitrarily direct the edges in $E(T_n) \setminus E(\overrightarrow{G})$. These edges have weight one. We prove that every target set D for $(\overrightarrow{G}, \tau)$ is also a target set in $(T_n, \tau' = 2n, \omega)$ and vice versa. The proof is similar to the proof of Proposition 2.1.

Let $D_0 \subseteq V(\overrightarrow{G})$ be a target set for $(\overrightarrow{G}, \tau)$ and $|D_0| = k$. Consider an activation process in $\overrightarrow{G}$ started from D_0 and let D_i be the set of vertices activated up to round i in the process. To show that D_0 is a target set for T_n, let $v \in V(T_n) \setminus D_0$ be an arbitrary vertex. We prove that if v is activated in $\overrightarrow{G}$ in a round, say i, then v as a vertex in T_n becomes active in round i. Let v be activated in $\overrightarrow{G}$ at round i. Hence, $|E^{in}(D_{i-1}, v)| \geq 2$, where $E^{in}(D_{i-1}, v)$ is the set of edges of $\overrightarrow{G}$ that enter from D_{i-1} to v. We obtain the following inequality which means v becomes active in T_n after activation of vertices in D_{i-1},

$$n|E^{in}(D_{i-1}, v)| \geq 2n = \tau'(v).$$

To prove the reverse direction, let $W_0 \subseteq V(T_n)$ be a target set for $(T_n, \tau' = 2, \omega)$ and $|W_0| = k$. Consider an activation process in T_n started from W_0, and let W_i be the set of vertices activated up to round i. We claim that W_0 is a target set for $\overrightarrow{G}$. Consider an activation process in $\overrightarrow{G}$ started from W_0 and denote by D_i the set of vertices activated up to round i in $\overrightarrow{G}$. We prove by induction on $0 < j \leq t$ that $D_j = W_j$, where t is the total number of activation rounds in T_n. In other words, we show that if a vertex v of T_n is activated in a round j, then it is activated in $\overrightarrow{G}$ in the same round. Suppose that $D_i = W_i$ for each $i < j \leq t$. Then $D_{j-1} = W_{j-1}$. Let $v \in W_j$ be an arbitrary vertex. We show that $v \in D_j$. Note that $|D_{j-1}| \leq n - 1$. Thus, in T_n, at most $n-2$ edges with weight 1 can enter from D_{j-1} to v. Also, since v is activated in T_n at round j then the following inequality holds, where $E^{in}(D_{j-1}, v)$ consists of the edges in $\overrightarrow{G}$ from D_{j-1} to v.

$$2n = \tau'(v) \leq n|E^{in}(D_{j-1}, v)| + (n - 2). \tag{4.1}$$

The latter inequality implies the following

$$\begin{aligned} 2 - |E^{in}(D_{j-1}, v)| &\leq \frac{n-2}{n} < 1 \\ 2 - |E^{in}(D_{j-1}, v)| &\leq 0 \\ 2 &\leq |E^{in}(D_{j-1}, v)|. \end{aligned}$$

Hence, v is activated in round j in $\overrightarrow{G}$. Similarly, by induction on j, we show that $D_j \subseteq W_j$. Let v be a vertex in D_j. Hence, $|E^{in}(D_{j-1}, v)| \geq 2 = \tau(v)$. Therefore

$$n|E^{in}(D_{j-1}, v)| \geq 2n = \tau'(v).$$

In other words, v is activated in $(T_n, \tau' = 2, \omega)$ in round j. Namely, $v \in W_j$. This completes the proof. □

5. Suggestions for further research

At the end of the paper, we propose some open problems for further research. We proved in this paper that TSSW(complete), TSSW(tournament) and TSSW(degenerate,complete) are NP-hard, where the edge-weights and threshold assignments are proportional to n, where K_n is the input complete graph or tournament. Do these problems remain NP-hard if either weights or thresholds or both of these data are constant (or limited) and do not depend on n, where the underlying graph is complete graph K_n or tournament $\overrightarrow{K_n}$? Perhaps some of these problems have polynomial time solutions for limited weights and thresholds.

Acknowledgments

The authors thank two anonymous reviewers of the paper for their useful comments.

References

[1] E. Ackerman, O. Ben-Zwi, G. Wolfovitz, *Combinatorial model and bounds for target set selection*, Theor. Comput. Sci. **411** (2010), 4017–4022.

[2] S. Askari, M. Zaker, *Spread of influence with incentives in edge-weighted graphs with emphasis on some families of graphs*, J. Combin. Optim. **47** (2024) No. 61.

[3] C.-L. Chang, Y.-D. Lyuu, *On irreversible dynamic monopolies in general graphs*, arXiv preprint, arXiv: (2009), arXiv:0904.2306v3.

[4] N. Chen, *On the approximability of influence in social networks*, SIAM Journal on Discrete Math. **23** (2009), 1400–1415.

[5] M. Chopin, A. Nichterlein, R. Niedermeier, M. Weller, *Constant thresholds can make target set selection tractable*, Theory of Computing Systems **55** (2014), 61–83.

[6] F. Cicalese, G. Cordasco, L. Gargano, K. Milanič, J. Peters, U. Vaccaro, *Spread of influence in weighted networks under time and budget constraints*, Theor. Comput. Sci. **586** (2015), 40–58.

[7] G. Cordasco, L. Gargano, M. Mecchia, A. A. Rescigno, U. Vaccaro, *Discovering small target sets in social networks: a fast and effective algorithm*, Algorithmica **80** (2018), 1804–1833.

[8] G. Cordasco, L. Gargano, J. G Peters, A. A. Rescigno, U. Vaccaro, *Fast and frugal targeting with incentives*, Theor. Comput. Sci. **812** (2020), 62–79.

[9] G. Cordasco, L. Gargano, A. A. Rescigno, U. Vaccaro, *Optimizing spread of influence in social networks via partial incentives*, International Colloquium on Structural Information and Communication Complexity, pages 119–134, Springer, 2015.

[10] E. D. Demaine, M. Hajiaghayi, H. Mahini, D. L. Malec, S. Raghavan, A. Sawant, M. Zadimoghadam, *How to influence people with partial incentives*, Proceedings of the 23rd international conference on World wide web, 2014.

[11] P.A. Dreyer, F. S. Roberts, *Irreversible k-threshold processes: graph-theoretical threshold models of the spread of disease and of opinion*, Discrete Applied Math. **157** (2009), 1615–1627.

[12] S. Ehard, D. Rautenbach, *On some tractable and hard instances for partial incentives and target set selection*, Discrete Optim. **34** (2019), 100547.

[13] U. Feige and Sh. Kogan, *Target set selection for conservative populations*, arXiv preprint, arXiv: (2019) 1909.03422.

[14] ______, *Target set selection for conservative populations*, Discrete Applied Math. **305** (2021), 119–132.

[15] P. Flocchini, R. Kralovic, A. Roncato, P. Ruzicka, N. Santoro, *On time versus size for monotone dynamic monopolies in regular topologies*, J. Discrete Algorithms **1** (2003), 129–150.

[16] D. Günneç, S. Raghavan, R. Zhang, *A branch-and-cut approach for the least cost influence problem on social networks*, Networks **76** (2020), 84–105.

[17] D. Günneç, S. Raghavan, R. Zhang, *Least-cost influence maximization on social networks*, INFORMS Journal on Computing **32** (2020), 289–302.

[18] K. Khoshkhah, H. Soltani, M. Zaker, *On dynamic monopolies of graphs: the average and strict majority thresholds*, Discrete Optim. **9** (2012), 77–83.

[19] ———, *Dynamic monopolies in directed graphs: spread of unilateral influence in social networks*, Discrete Applied Math. **171** (2014), 81–89.

[20] K. Khoshkhah, M. Zaker, *On the largest dynamic monopolies of graphs with a given average threshold*, Canadian Mathematical Bulletin **58** (2015), 306–316.

[21] R. S. Melo, A. L. Vignatti, F. K. Miyazawa, M. J. Ota, *Combinatorial dual bounds on the least cost influence problem*, Pesquisa Operacional **43** (2023), e275468.

[22] S. Raghavan, R. Zhang, *Weighted target set selection on trees and cycles*, Networks **77** (2021), 587–609.

[23] M. Zaker, *On dynamic monopolies of graphs with general thresholds*, Discrete Math. **312** (2012), 1136–1143.

[24] ———, *Generalized degeneracy, dynamic monopolies and maximum degenerate subgraphs*, Discrete Applied Math. **161** (2013), 2716–2723.

[25] ———, *On irreversible spread of influence in edge-weighted graphs*, Contributions to Discrete Mathematics **15** (2020), 60–72.

Department of Mathematics,
Institute for Advanced Studies in Basic Sciences,
Zanjan 45137-66731, Iran
E-mail address: s.askari@iasbs.ac.ir

Department of Mathematics,
Institute for Advanced Studies in Basic Sciences,
Zanjan 45137-66731, Iran
E-mail address: mzaker@iasbs.ac.ir