Contributions to Discrete Mathematics

Volume 20, Number 2, Pages 497–507
ISSN 1715-0868

FIBONACCI NUMBERS AND n-COLOUR COMPOSITIONS

RACHNA SACHDEVA, BHARTI GUPTA, AND URVASHI SHARMA

Abstract. Agarwal defined n-colour compositions as ordered n-colour partitions in 2000. In this paper, we relate n-colour compositions with a problem of sewage water treatment using a bijection which involves the well-known Fibonacci numbers. In addition, we study a new restricted n-colour composition function combinatorially. Our work also includes a one-to-one correspondence between restricted n-colour compositions and the number of ways to tile a particular type of rectangle using dominoes and squares.

1. Introduction

A composition of a positive integer n is an ordered representation of n as a sum of positive integers. For example, there are 4 compositions of 3 viz $3, 1+2, 2+1, 1+1+1$, whereas a partition is an unordered representation of n as a sum of positive integers, i.e. $1+2$ and $2+1$ are the same partitions but different compositions. The parts of a partition or composition are known as summands. MacMahon in [16] has given a detailed study of compositions. The close relationship between compositions and Fibonacci numbers is given by Cayley [6] and Stanley [23]. First we recall the following definitions:

Definition 1.1. *The Fibonacci numbers F_n are defined as*

$$F_0 = 1, F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2},\ n \geq 2.$$

The following theorem appears in several places in the literature. The earliest reference can be found in [23].

Theorem 1.2. *The number of compositions of n into odd parts equals F_n.*

Cayley proved the following theorem in [6]:

Received by the editors June 18, 2024, and in revised form September 29, 2024.
2000 *Mathematics Subject Classification.* 05A15, 05A17, 05A19, 11P81.
Key words and phrases. n-colour partitions, n-colour compositions, Fibonacci numbers, sewage water treatment, tilings, dominoes.
The authors are thankful to the anonymous referee for his/her valuable suggestions, which led to a better presentation of this paper.

Theorem 1.3. *The number of compositions of $n+1$ into parts greater than one equals F_n.*

Many other results related to compositions of integers are given in [14, 15]. Analogous to partitions of integers, A. K. Agarwal defined n-colour partitions in [1]. Agarwal also defined n-colour compositions in [2]. Agarwal, along with other authors, proved many combinatorial properties of these new objects (see [3, 18, 19]).

Definition 1.4. *A partition with "n copies of n" (or an n-colour partition) is a partition in which a part of size n, $n \geq 1$, can occur in n different colours. These colours are denoted by subscripts and the parts satisfy the order*

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < 5_2 < \cdots$$

For example, the integer 3 has 3 colours viz. $3_1, 3_2, 3_3$. There are 6 n-colour partitions of 3 viz. $3_1, 3_2, 3_3, 2_1 + 1_1, 2_2 + 1_1, 1_1 + 1_1 + 1_1$.

Definition 1.5. *An ordered n-colour partition is an n-colour composition.*

For example, n-colour compositions of 3 are $3_1, 3_2, 3_3, 2_1+1_1, 2_2+1_1, 1_1+2_1, 1_1+2_2, 1_1+1_1+1_1$.

Suppose the number of n-colour partitions and compositions of an integer ν are denoted by $P(\nu)$ and $C(\nu)$ respectively. The generating functions for $P(\nu)$ and $C(\nu)$ are given in [4] and [2] respectively as follows:

$$\sum_{n=0}^{\infty} P(\nu)q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n},$$

$$\sum_{n=0}^{\infty} C(\nu)q^{\nu} = \frac{q}{1-3q+q^2}.$$

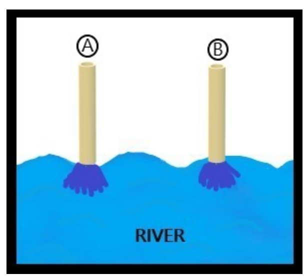
It was also proved in [2] that the number of n-colour compositions of ν is the same as 2νth Fibonacci number i.e. $C(\nu) = F_{2\nu}$. Much progress has been made in the field of n-colour compositions since they were introduced in [2]. For example, restricted n-colour compositions have been studied by several authors ([10, 11, 21], for instance). Analogous to self-inverse compositions defined by MacMahon, n-colour self-inverse compositions have been defined and studied combinatorially in [17, 20]. n-colour compositions are important for the study of part products and palindromes (see, for instance [22]). Various graphical representations for n-colour compositions are given in [19, 21].

In this paper, we give an application of n-colour compositions to a real-life problem related to sewage water treatment along with some other results. In Section 2, we provide a bijective proof of our main result related to sewage water treatment. In Section 3, we study a restricted n-colour composition function and relate it to a combinatorial problem. In the concluding section, we give some direction for future work.

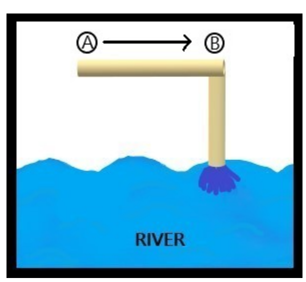
2. n-colour compositions and Sewage treatment problem

The relationship between Fibonacci numbers and sewage treatment plants was first studied by R. A. Deininger in [8]. Suppose there are n towns on the bank of a river. These towns discharge their untreated sewage into the river and pollute the water. The towns would like to build a treatment plant to control the pollution. It is economical to build one or more central treatment plants along the main sewer and then send the wastewater from each town to another one. It is not financially beneficial to split the sewage of a town between two adjacent towns since this would require the building of two sewers for the same town. Thus it is clear that every town must have its sewage treated somewhere and at least one town must discharge the cleaned water into the river. Let $S(n)$ denote the number of economical solutions. Clearly $S(1) = 1$. For $n = 2$, we have two towns say A and B. In this case, there are only three possible solutions:

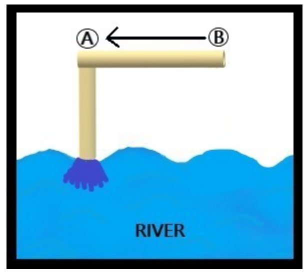
(1) Each town can treat their own sewage and discharge clean water into the river.



(2) Town A can send its sewage along the pipe (to the right) to B for treatment and B discharges it into the river.



(3) Town B can send its sewage to A (to the left) which treats it with its own dirty water and discharges it into the river.



Thus each town can build its own treatment plant, send the sewage upstream ($\longrightarrow$), or send it downstream ($\longleftarrow$).

Let 0 denote no transport between neighbouring towns, 1 denote upstream transport and 2 denote downstream transport. In this way, we get $S(2) = 3$. In other words, the only possibilities for $n = 2$ are 0,1,2. For $n = 3$, the possibilities are $00, 01, 02, 10, 11, 12, 20, 21,$ and 22, where 21 is not a solution, since a town cannot simultaneously transfer waste both upstream and downstream. In this way, we get

$$S(3) = 8 = 3S(2) - S(1).$$

Next, $n = 4$ gives $S(4) = 21$ solutions given as follows:

000	001	002	010	011	012	020	022
100	101	102	110	111	112	120	122
200	201	202	~~210~~	~~211~~	~~212~~	220	222

Notice that $S(4) = 3S(3) - S(2)$ since there are three words containing 21. In general, consider $n + 1$ towns with $S(n + 1)$ solutions. Adding one town increases the number of solutions to $3S(n)$. From this we must subtract the number of solutions containing 21, namely $S(n - 1)$. This gives us the general relation

$$S(n + 1) = 3S(n) - S(n - 1), \; n \geq 1. \tag{2.1}$$

Using this recurrence relation, the value of $S(n)$ can be computed for various values of n. It was also proved in [8] that $S(n)$ equals F_{2n}, the $2n$th Fibonacci number.

We have the following theorem which is proved in two ways: a direct proof and a bijective proof.

Theorem 2.1. *For $\nu \geq 1$, $S(\nu) = C(\nu)$.*

Direct proof. It is proved in [21] that $C(\nu)$ satisfies the same initial conditions and recurrence relation (2.1) satisfied by $S(\nu)$. In this way, we get $S(\nu) = C(\nu)$. □

Bijective Proof. Let Ω_ν denote the set of solutions enumerated by $S(\nu)$ and Δ_ν denote the set of n-colour compositions enumerated by $C(\nu)$. Let α be a solution belonging to the set Ω_ν. Suppose there is sewage transportation of sewage between k towns and water is discharged from the mth town, where

$1 \leq k \leq \nu,\ 1 \leq m \leq k$. Then it corresponds to an n-colour part k_m of an n-colour composition. Note that $k = 1, m = 1$ is the case when a town has its own sewage treatment plant and there is no transportation. This will correspond to the n-colour part 1_1. Since the total number of towns is ν, we get that the resulting n-colour composition belongs to the set Δ_ν.

Conversely, suppose β is an n-colour composition belonging to the set Δ_ν. Let r_s denote a summand in β. This corresponds to a transport of sewage between r number of towns and water is discharged from the sth town. Since β is an n-colour composition of ν, the sum of all summands is equal to ν. Hence, the corresponding solution belongs to Ω_ν. □

To illustrate the bijection we have constructed to prove Theorem 2.1, we give the example for $\nu = 3$ shown in Figure 1.

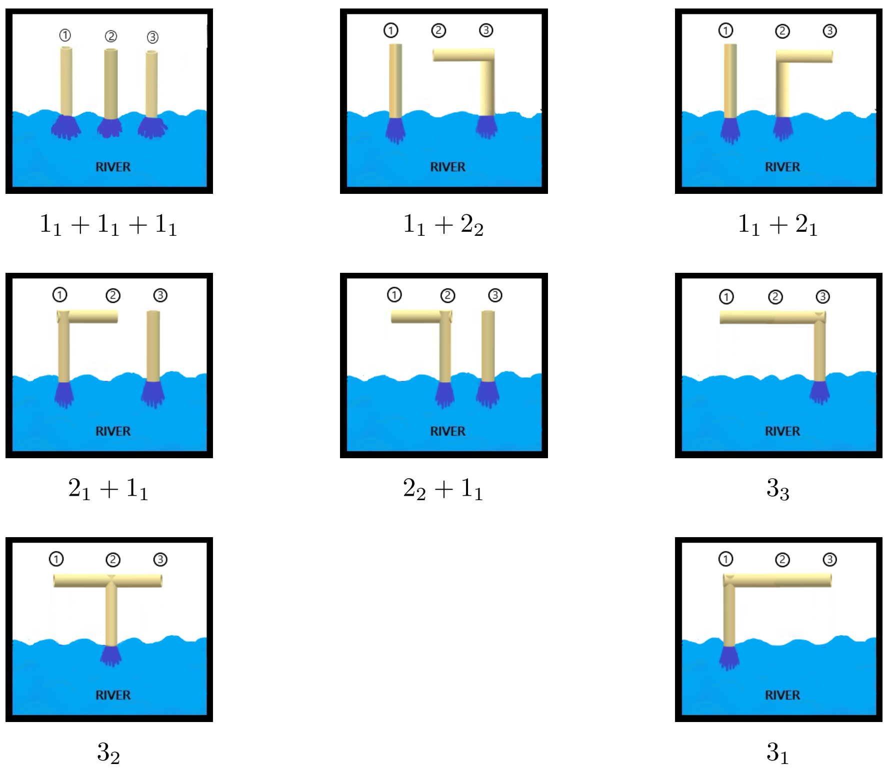


FIGURE 1. Illustration of the bijection to prove $C(3) = S(3)$. All illustrations are the corresponding solutions enumerated by $S(3)$, and below them are the n-colour compositions enumerated by $C(3)$.

3. A RESTRICTED n-COLOUR COMPOSITION FUNCTION

In this section, we restrict the parts of n-colour compositions to 1 and 2. Let $A(\nu)$ denote the number of n-colour compositions of ν such that only 1 and 2 are allowed as parts and $A(m,\nu)$ denote the number of n-colour compositions enumerated by $A(\nu)$ into m parts. Let $A(q)$ and $A(m;q)$ denote the enumerative generating functions for $A(\nu)$ and $A(m,\nu)$, respectively. We have obtained the following theorem:

Theorem 3.1.

$$A(m;q) = (q+2q^2)^m, \tag{3.1}$$

$$A(q) = \frac{q+2q^2}{1-q-2q^2}, \tag{3.2}$$

$$A(\nu) = A(\nu-1) + 2(\nu-2),\ \ \nu \geq 2, \tag{3.3}$$

$$A(0) = 1, A(1) = 1.$$

Proof.

$$\begin{aligned} A(m;q) =& \sum_{\nu=1}^{\infty} A(m,\nu)q^{\nu} \\ =&(q+2q^2)^m. \end{aligned}$$

$$\begin{aligned} A(q) &= \sum_{m=1}^{\infty} A(m;q) \\ &= \sum_{m=0}^{\infty} (q+2q^2)^m - 1 \\ &= \frac{1}{1-(q+2q^2)} - 1 \\ &= \frac{q+2q^2}{1-q-2q^2}. \end{aligned}$$

Recurrence (3.3) can be proved by using the generating function (3.2) but here we give a combinatorial proof as follows:

We split the n-colour compositions enumerated by $A(\nu)$ into two classes:

I. Those that have 1_1 as the extreme left part.
II. Those that have 2_1 or 2_2 as the extreme left part.

We transform the n-colour compositions in each case as follows:

In class I, we delete the part 1_1 and get a composition enumerated by $A(\nu-1)$. Clearly this process is reversible and hence the number of n-colour compositions in class I is $A(\nu-1)$.

Next, we transform the n-colour compositions in class II by deleting the extreme left part 2_1 or 2_2 as the case may be. In this way, we get a composition enumerated by $A(\nu-2)$. Again, this process can be reversed and we get that the number of n-colour compositions in class II is $2A(\nu-2)$.

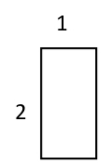
The above transformations clearly establish a bijection between the n-colour compositions enumerated by $A(\nu)$ and those enumerated by $A(\nu-1)+2A(\nu-2)$. Thus recurrence (3.3) is proved. □

We connect the sequence $A(\nu)$ to a problem of counting ways to tile a particular type of rectangle. The problem of counting the number of ways to tile an $m \times n$ rectangle has been studied by several authors (see, for instance [5, 7, 12, 13]). Various authors have used $k \times k$ squares, dominoes, L-shaped trominos, etc., for this purpose. In this paper, we consider a particular case when $m = 2$. Let $T(n)$ denote the number of ways of tiling a $2 \times n$ rectangle using 2×2 squares and 2×1 dominoes (each domino of size 2×1 may be rotated). The following theorem can be easily obtained:

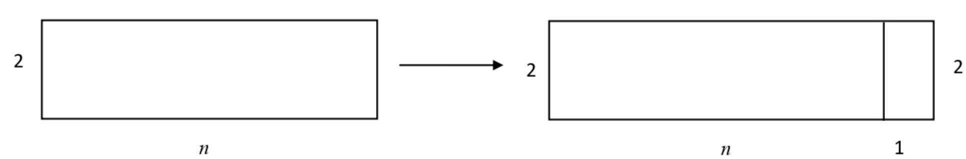
Theorem 3.2.

$$\begin{aligned} &T(n) = T(n-1) + 2T(n-2), \quad n \geq 2, \\ &T(0) = 1,\ T(1) = 1. \end{aligned}$$

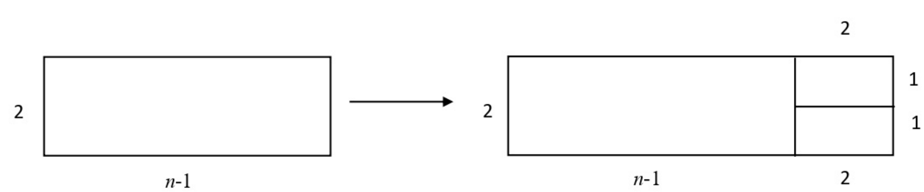
Proof. For $n = 0$, there is only one empty rectangle.
For $n = 1$, there is only one way to tile a 2×1 rectangle by using a 2×1 domino.



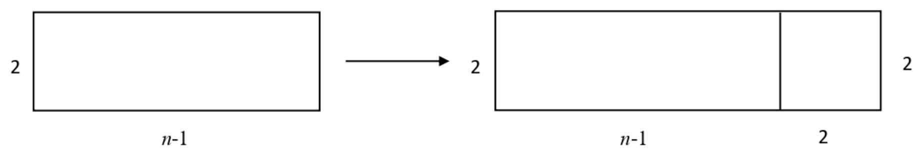
Suppose we have tiled up to $2 \times n$ rectangles. Now we can tile a $2 \times (n+1)$ rectangle by using $2 \times n$ and $2 \times (n-1)$ rectangles in the following ways:
(I) Add a 2×1 domino on the right of $2 \times n$ rectangle



(II) Add two 2×1 dominoes by rotating them to the right of a $2 \times (n-1)$ rectangle



(III) Add a 2×2 square to the right of a $2 \times (n-1)$ rectangle



In this way, we have obtained all the ways to tile a $2 \times (n+1)$ rectangle and we see that it satisfies the following recursive relation:

$$T(n) = T(n-1) + 2T(n-2), \qquad n \geq 2,$$
$$T(0) = 1, \; T(1) = 1.$$

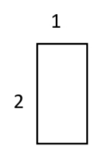
□

Theorem 3.3. *$A(\nu) = T(\nu)$ for all $\nu \geq 1$.*

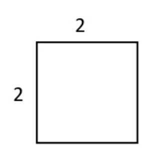
Proof. Both $A(\nu)$ and $T(\nu)$ satisfy same initial conditions and recurrence relation. Here we give the bijective proof as follows:

Let Θ_ν denote the set of n-colour compositions enumerated by $A(\nu)$ and Σ_ν denote the set of $2 \times \nu$ rectangles enumerated by $T(\nu)$. We establish a bijection between Θ_ν and Σ_ν by mapping each composition in Θ_ν to a $2 \times \nu$ rectangle in Σ_ν. We do this by mapping each part of a composition in Θ_ν to either a 2×2 square or a 2×1 domino. Let $\pi = (i_1)_{j_1} + (i_2)_{j_2} + \cdots + (i_r)_{j_r}$ be an n-colour composition in Θ_ν. We define $\psi : \Theta_\nu \to \Sigma_\nu$ on the parts $(i_t)_{j_t}, 1 \leq t \leq r$ of π. Since 1 and 2 are the only allowed parts, this gives $1 \leq j_t \leq i_t \leq 2$, for any $1 \leq t \leq r$. Thus we have the following cases:

Case I: $i_t = 1$, $j_t = 1$. In this case, we map the part 1_1 to a 2×1 domino:

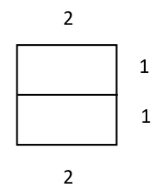
$\psi(1_1) =$ 

Case II: $i_t = 2, \; j_t = 1$. The part 2_1 is mapped to a 2×2 square:

$\psi(2_1) =$ 

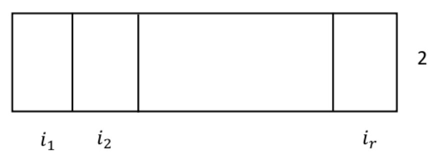
Case III: $i_t = 2, \; j_t = 2$. We map the part 2_2 to two 2×1 dominoes joined together after rotating them in the following way:

$$\psi(2_2) =$$



Now, the map ψ is defined on the composition π as follows:

$$\psi(\pi) = \psi((i_1)_{j_1} + (i_2)_{j_2} + \cdots + (i_r)_{j_r}) =$$



Since $(i_1)_{j_1} + (i_2)_{j_2} + \cdots + (i_r)_{j_r} = \nu$, the 2×2 squares and 2×1 dominoes joined together will give us a $2 \times \nu$ rectangle which belongs to Σ_ν. In this manner, π corresponds to a unique $2 \times \nu$ rectangle which is tiled by using 2×2 squares and 2×1 dominoes. This process can be reversed easily to get an n-colour composition in Θ_ν from a $2 \times \nu$ rectangle in Σ_ν. This establishes a one-to-one correspondence between the two. $\square$

To illustrate the bijection we have constructed to prove Theorems 3.3, we give the example for $\nu = 3$ shown in Figure 2.

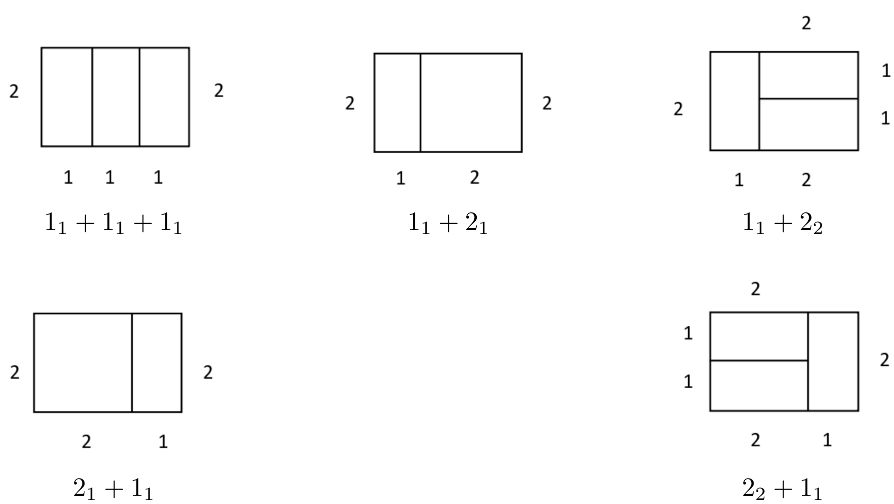


FIGURE 2. Illustration of the bijection to prove $A(3) = T(3)$. Each image shows a 2×3 rectangle enumerated by $T(3)$, labeled below with the corresponding n-colour composition enumerated by $A(3)$.

4. Conclusion

In this paper, we have associated a restricted n-colour composition function with the solution of a restricted tiling problem. It is natural to ask whether unrestricted n-colour compositions can be related to more general tiling problems. The answer to this question will be of great help to establish new connections between tiling problems and the Fibonacci numbers.

References

[1] A. K. Agarwal, *Partitions with "N copies Of N"*, Lecture Notes in Math., **No. 1234**, Proceedings of the Colloque de Combinatoire Enumerative, University of Quebec at Montreal, Springer-Verlag, 1985, pp. 1–4.

[2] ———, *n-colour compositions*, Indian J. Pure Appl. Math, **31**(11)(2000), 1421–1427.

[3] ———, *An analogue of Euler's identity and new combinatorial properties of n-colour compositions*, J. Comput. Appl. Math., **160**(2003), 9–15.

[4] A. K. Agarwal and G. E. Andrews, *Rogers–Ramanujan identities for partitions with "N copies of N"*, J. Combin. Theory Ser. A, **45**(1), (1987), 40–49.

[5] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, MAA, 2003.

[6] A. Cayley, *Theorems in trigonometry and on partitions*, Collected Mathematical Papers, vol. 10, 16.

[7] P. Z. Chinn and D. Oliver, *Some results inspired by covering rectangles with* 1×1 *and* 1×3 *rectangles*, Congr. Numer., 122 (1996), 119-124.

[8] R. A. Deininger *Fibonacci numbers and water pollution control*, Fibonacci Q., Vol 10, no. 3 (1971), 299–302.

[9] R. P. Grimaldi, *Fibonacci and Catalan Numbers*, John Wiley & Sons, Inc., 2012.

[10] Y.-H. Guo, *n-colour even compositions*, Ars Combin. **109**(2013), 425–432.

[11] ———, *Some n-colour compositions*, J. Integer Seq., **15**(2012), Article 12.1.2, 10pp.

[12] S. Heubach, *Tiling an m-by-n area with squares of size up to k-by-k with* $m \leq 5$, Congr. Numer., **140**(1999), 43–64.

[13] R. Hochberg and M. Reid, *Tiling with notched cubes*, Discrete Math., **214**(2000), 255–261.

[14] P. A. MacMahon, *Memoir on the theory of compositions of numbers*, Phil. Trans., **185**(1894), 835–901.

[15] ———, *Second memoir on the compositions of numbers*, Phil. Trans., **207**(1908), 65–135.

[16] ———, *Combinatory Analysis*, AMS Chelsea Publishing, New York, Vol I, II(2001).

[17] G. Narang and A. K. Agarwal, *n-colour self-inverse compositions*, Proc. Indian Acad. Sci(Math. Sci.), **116**(3)(2006), 257–266.

[18] ———, *Some combinatorial properties of n-colour compositions*, South East Asian J. Math. & Math. Sc.,**6**(2), (2008), 37–43.

[19] ———, *Lattice Paths and n-colour compositions*, Discrete Math., **308**(9), (2008), 1732–1740.

[20] R. Sachdeva, *On n-colour self-inverse compositions*, J. Ramanujan Math. Soc., **39**(1), (2024), 39–53.

[21] R. Sachdeva and A. K. Agarwal, *Combinatorics of certain restricted n-colour composition functions*, Discrete Math., **340**(2017), 361–372.

[22] C. Shapcott, *C-colour compositions and palindromes*, Fibonacci Q. **50**(2012), no.4, 297–303.

[23] R. P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth, 1986; reissued, Cambridge, 1999.

Department of Mathematics, Guru Nanak Dev University, Amritsar-143005, India
E-mail address: rachna.sachdeva.1989@gmail.com

Department of Mathematics, Doaba College, Jalandhar-144001, India
E-mail address: bhartigupta.maths@gmail.com

Department of Mathematics, Doaba College, Jalandhar-144001, India
E-mail address: urvashi433@gmail.com