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John Stillwell, *Elements of Mathematics: From Euclid to Gödel*. Princeton: Princeton University Press, 2016. Pp. xiv + 422, illus. US $39.95 (cloth).
ISBN: 9780691171685

Reviewed by Brenda Davison, Simon Fraser University

Elements of Mathematics, from Euclid to Gödel is the latest book from John Stillwell and what a book it is. It covers a vast swath of mathematics in its 422 pages with an impressive degree of rigour, particularly given the intended audience. To set expectations (correctly as it turns out), Stillwell quotes mathematician G.H Hardy on the first page: "a book on mathematics without difficulties would be worthless" (xi).

Stillwell has divided mathematics into elementary and advanced topics and has structured the book with chapters on elementary topics in each of the following subjects: arithmetic, computation, algebra, geometry, calculus, combinatorics, probability, and logic. The final chapter presents an advanced topic from each of the listed subjects. The line between elementary and advanced mathematics is blurry – infinity, abstraction and proof become more prominent as the mathematics moves from elementary to advanced. And, as Stillwell points out, you have to "occasionally cross the line, in order to make it clear where the line lies" (194).

Infinity is the most important characteristic that delineates an advanced topic from an elementary one but how much infinity is elementary? An elementary topic, volume, is shown to be deeper than area by way of an example that it is not possible to convert a regular tetrahedron to a cube with a finite number of additions and subtractions (158). By page 33, Stillwell has introduced potential and actual infinity. With the natural numbers, a potential infinity, and the real numbers, an actual or completed infinity, he draws a rough line between an elementary topic, which may employ a potential infinity, and an advanced topic, which may make use of a completed infinity. This distinction between elementary and advanced is highlighted and remarked on throughout the text and is one of a number of interesting threads that run throughout the book and provide cohesion.

Consistently, Stillwell speaks to the reader as an intelligent, curious person who he expects to understand the material. His love the subject material shines brightly throughout. This makes the book readable — in fact, compelling. Perhaps the best example of this is in the chapter on logic. A thought-provoking discussion is about reverse mathematics and the depth of theorems as measured by the axioms that are needed to prove them. In fact, this is an intriguing and interesting choice to include in a survey book; reverse mathematics is a new field founded in 1975 and for which the definitive book was written in 2009.[1]

[1] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, 5th edition. Perspectives in Logic. (Cambridge: Cambridge University Press; Cambridge: Association for Symbolic Logic, Poughkeepsie, NY).

Each chapter ends with both historical and philosophical remarks, a delightful addition to a book from which a lot of mathematics can be learned. I was thrilled to note references to recent historical research — for example Plofker's 2009 work on ancient Indian mathematicians[2] (236). These sections allow Stillwell to demonstrate his love of mathematics and a sharp wit (for example: "educated people in the 1700s could be abysmally ignorant of mathematics (some things never change…" [29]). Most of the stated theorems are proved and, typically, in an extraordinarily clear way — I agree with Stillwell that postponing proof until the later stages of an undergraduate education and that examples of proof being part of mathematics from the high school level is delusional. Other pedagogical comments are well-placed and in our current era of computational power we would be well-advised to note Stillwell's advice that "the new trio of computation-combinatorics-logic now deserves to be taken as seriously as the old trio of arithmetic-algebra-geometry" (2).

Often the proofs are geometric or are accompanied by very clear drawings or figures. The proof that $\sqrt{2}\sqrt{3} = \sqrt{6}$ (157) or the demonstration that $\ln(ab) = \ln(a) + \ln(b)$ (208) or the proof of the irrationality of e (225) are of the best type. They do not just verify the claim but give insight and clarity into why the statement or theorem must be true. Stillwell has used a variety of proof techniques throughout and is careful to explain how each technique is used and why each technique is valid before he uses it for the first time. A student of this book could learn much about proof by a careful reading of these explanations and the accompanying proofs.

The jacket cover (with its beautiful cover art built upon Fra Carnevale's *The Ideal City*) makes the ambitious claim that readers of *Elements of Mathematics* can range from high school students through to professional mathematicians. I would expect a high school student would need a competent guide as he or she read this book. I would, however, recommend that every high school mathematics classroom have a copy of this book — a perfect resource for the keen student, a ready answer to the questions of what mathematics is or why we study it, and a ready guide to show glimpses of the forest rather than just a few of the trees.

Stillwell, in fact, brilliantly tells the reader clearly, explicitly, and early in the book that mathematics has changed from acquiring skill at solving specific problems to a study of structure. He claims mathematics can be viewed from a higher standpoint – that of "structure and axiomatization, which identifies certain algebraic laws and classifies algebraic systems by the laws they satisfy" (7). Under this view, the integers with the operations of addition and multiplication are an example of a more abstract idea, a ring, and the real numbers, again with addition and multiplication, are an example of a different structure, a field. Many other examples of rings and fields of course exist — high school mathematics' view is to first see one specific tree (the real numbers) in a very large forest (a field). This higher viewpoint idea is also a theme that runs throughout the book allowing the reader to see the subject matter more as a unified whole.

The cover art is cleverly used as an example in the section on projective geometry (354), and three other artworks are used in the section on affine geometry (387-389). These well-chosen examples clearly show affine geometry done well, done poorly, and done with subtle failure. Like the illustrations in the book, these additions provide a lot of insight into what the mathematics is telling us.

I do not see that this book could be used as a textbook, partially due to the fact that it includes no exercises. It occasionally uses non-standard terminology although it is always footnoted (this is the closest I can come to a critical comment). Book recommendations for various topics — mathematical, historical, and philosophical — are judiciously positioned throughout the book with the inclusion of an extensive bibliography. This book is a perfect base from which to explore what mathematics has to offer — the book

[2] Kim Plofker, *Mathematics in India*. (Princeton, NJ: Princeton University Press, 2009).

to read and reread starting in high school and certainly continuing through an undergraduate mathematics degree. In short, I recommend it unreservedly.